國立清華大學 107 學年度碩士班考試入學試題

系所班組別：科技管理研究所(0548)

考試科目（代碼）：4801 微積分

共_1_頁，第_1_頁　　　*請在【答案卷、卡】作答

第一題填充部分請將答案依題號順序寫在答案卷上，不必寫演算過程。第二題至第六題必須詳細寫出計算及證明過程，否則不予計分。

1. Fill in the blank with your answer (7 pts each)
   (a) Find the maximum value of $f(x) = x^3 \ln(1/x)$. Ans: = _______.
   (b) Find the centroid of the region between the x-axis and the arch $y = \sin x$, $0 \le x \le \pi$. Ans: = _______.
   (c) Find the area of the elliptical region cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$. Ans: = _______.
   (d) Let $a > b > c > 0$. Find the limit: $\lim_{n\to\infty} (a^n + b^n + c^n)^{1/n}$. Ans: = _______.
   (e) Suppose $e^x = \sum_{n=0}^{\infty} a_n (x-2)^n$. Then the coefficient a_{10} = _______.
   (f) Under the conditions $x > 0$, $y > 0$, $z > 0$ and $x^2 + y^2 + z = 32$, the maximum value of the product xyz is _______.

2. (10 pts) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series, where $a_n \ge 0$. Does the series $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ have to be convergent? Give reasons for your answer.

3. (10 pts) Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) = x^6 + x^3 - x$ for all $x \in \mathbb{R}$? Give reasons for your answer.

4. (12 pts) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that there exists a point c in $[a, b]$ such that

$$\int_a^c f(x)\,dx = \int_c^b f(x)\,dx.$$

(Hint: Use the intermediate value theorem)

5. (12 pts) Solve the integral equation

$$y(x) + \int_0^x y(t)\,dt = x.$$

6. (14 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $0 \le f'(x) \le f(x)$ for all $x \in \mathbb{R}$. Show that $g(x) = e^{-x} f(x)$ is decreasing. If f vanishes at some point, conclude that f is identically zero.