Total: 100 points.

1. (7 points each) Fill in your answers

   - $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \underline{\quad (a) \quad}$.

   - The interval of convergence of $\sum_{k=0}^{\infty} k! \, (2x-1)^k$ is $\underline{\quad (b) \quad}$.

   - The volume enclosed by $x=0, y=0, z=0$, and $2x+y+3z=6$ is $\underline{\quad (c) \quad}$.

   - Use the Newton-Raphson method to estimate $\sqrt{2}$. The iterative scheme is $\underline{\quad (d) \quad}$. The approximation with a precision to two decimal places is $\underline{\quad (e) \quad}$.

   - When the point set (x, y) moves from $(1, 2)$ to $(1.05, 2.01)$, the increments dz and Δz for $z = x^2 + 5y^2$ are $\underline{\quad (f) \quad}$ and $\underline{\quad (g) \quad}$, respectively.

2. (15 points) Show your work to calculate the limit of

$$\frac{\sqrt{2\pi} - \int_{-\infty}^{x} e^{-z^2/2} dz}{x^{-\alpha} e^{-x^2/2}}$$

   when $x \to \infty$ for each real value α.

3. (a) (5 points) State the Mean-Value Theorem (single variable).

   (b) (10 points) Let h be a real-valued function and twice differentiable on the real numbers. Prove that for any $a < b$,

$$\frac{\partial}{\partial x} \int_a^b h(xz) dz = \int_a^b h'(xz) \, z \, dz.$$

4. A differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}$$

   acting on a real-valued function $P(t, x; \sigma)$ is defined by

$$\mathcal{L}P(t, x; \sigma) = \frac{\partial P}{\partial t}(t, x; \sigma) + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial x^2}(t, x; \sigma), \tag{1}$$

   where $t \in [0, \infty)$ and $x \in (-\infty, \infty)$ are variables, and $\sigma > 0$ is a parameter. Assume that $P(t, x; \sigma)$ is smooth [that means you can take partial derivatives of any order on P] and $P(t, x; \sigma)$ solves (1) [that means $\mathcal{L}P(t, x; \sigma) = 0$].

   (a) (7 points) Check that for any positive integer n, $\frac{\partial^n}{\partial x^n} P(t, x; \sigma)$ solves (1) as well.

(b) (7 points) For any real number α, $T > t$ and any positive integer n, define

$$Q(t,x;\sigma) = \alpha\,(T-t)\,\frac{\partial^n}{\partial x^n}P(t,x;\sigma).$$

Check that it satisfies

$$\mathcal{L}Q(t,x;\sigma) = -\alpha\,\frac{\partial^n}{\partial x^n}P(t,x;\sigma).$$

(c) (7 points) Prove that

$$\frac{\partial P}{\partial \sigma}(t,x;\sigma) = \sigma\,(T-t)\,\frac{\partial^2 P}{\partial x^2}(t,x;\sigma).$$

[Hint: you may use $\frac{\partial}{\partial \sigma}(\mathcal{L}P) = 0$.]