國 立 清 華 大 學 命 題 紙

96學年度 計量財務金融學系（所） 乙組 碩士班入學考試

科目 基礎數學（微積分、線性代數） 科目代碼 5304 共 2 頁第 1 頁 *請在【答案卷卡】內作答

Total Points: 100.

1. (10 %) Given $\varphi_n(u) = \left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)^n$ and $\varphi(u) = e^{\frac{1}{2}u^2}$ with $u \in \Re$(the set of all real numbers) and $n > 0$, prove that for any $u \in \Re$,

$$\lim_{n\to\infty} \varphi_n(u) = \varphi(u).$$

2. (15 %) Given a real number $M > 0$, prove that

$$\int_M^\infty e^{-x^2/2}dx \leq \sqrt{\frac{\pi}{2}}e^{-\frac{M^2}{2}}.$$

(Hint: you may double the left hand side and consider a new integral domain $\{(x,y)|x,y \in [M,\infty)\} \subseteq \{(x,y)|x^2+y^2 \geq M^2, x \geq 0, y \geq 0.\})$

3. Given the following one dimensional ordinary differential equation: for $x \geq 0$

$$\begin{cases} \frac{dy}{dx} & = \alpha(m - y(x)) \\ y(0) & = y_0 \end{cases}$$

where α, m, and y_0 are arbitrary real numbers.

   (a) (7 %) Solve $y(x)$. (Hint: let $z(x) = m - y(x)$.)

   (b) (8 %) Find all conditions such that the limiting solution of $y(x)$ is equal to m as $x \to \infty$.

4. (15 %) Let $f(x,y) = xy^2 + x + 2y$ be the cost function and $g(x,y) = xy - 1$ be the constraint function. Find the minimum and maximum, if they exist, of the cost function subject to the constraint $g(x,y) = 0$ with $x > 0$. In the case that they do exist, identify all of the points (x,y) at which these values are attained.

5. Given a matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and a vector $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$,

   (a) (5 %) Is $Ax = b$ solvable? Explain your result.

   (b) (5 %) Find the least squares solution of $Ax = b$. (Denote this solution by $\hat{x}$.)

(c) (5 %) Show that $e = b - A\hat{x}$ is orthogonal to the column space of A.

(d) (5 %) Give a geometric interpretation of $A\hat{x}$.

6. Given a Markov matrix $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$,

   (a) (4 %) Find the eigenvalues of A.

   (b) (9 %) If $A^{\infty} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $x_1 + x_2 = 1$, Find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

7. True or False

   (a) (3 %) If the matrix A^T is invertible if and only if A is invertible.

   (b) (3 %) If $A^T A$ is invertible, then A is invertible.

   (c) (3 %) Let $f \geq g \geq 0$ be real-valued and continuous functions defined on $[0,1]$. If $\int_0^1 f(x)dx = 0$, then $g(x) = 0$, $\forall x \in [0,1]$.

   (d) (3 %) If $\sum_{n=0}^{\infty} a_i(n)$ is divergent and each a_i is differentiable, then $\sum_{n=0}^{\infty} a_i'(n)$ is divergent.