注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 108 學年度碩士班考試入學試題

系所班組別：**計量財務金融學系 乙組**

考試科目(代碼)：**微積分(5104)**

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 作答中如有發現試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 108 學年度碩士班考試入學試題

系所班組別：計量財務金融學系碩士班 乙組

考試科目（代碼）：微積分 （5104）

 *請在【答案卷】作答

Total Points: 100. Calculation and Proofs. Show your derivation.

1. (10 pts) Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \le 4$.

2. (24 pts) Given $x \ge 0$, k a constant, and $y^+ = max\{0, y\}$, prove each inequality (three inequalities)

$$x - k \le (x-k)^+ \le \int_{-\infty}^{\infty} (xe^{\frac{-1}{2}+z} - k)^+ \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz \le x.$$

3.(24 pts) On a domain $[0,T]$ with the partition $\Pi = \{t_0 = 0, t_1, \cdots, t_n = T > 0\}$ and its length $\|\Pi\| = max_{i=0,\cdots,n-1}(t_{j+1} - t_j)$, two definitions of variation are given below

(1) The total variation of f, denoted by $TV_T(f)$ is

$$TV_T(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|.$$

(2) The quadratic variation of f, denoted by $\langle f, f \rangle_T$ is

$$\langle f, f \rangle_T = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2.$$

Now, suppose $f \in C^1([0,T])$ meaning that its first-order derivative is continuous on $[0,T]$. Answer the following questions:

(a) State the mean value theorem.

(b) Apply this theorem to prove $TV_T(f) = \int_0^T |f'(t)|\, dt$.

(c) Prove $\langle f, f \rangle_T = 0$.

4.(24 pts) Given two sets of probabilities $P = \{p_i, i = 1, \dots, n\}$ and $Q = \{q_i, i = 1, \dots, n\}$, the relative entropy between P and Q is

$$\mathcal{E}(P|Q) = \sum_{i=1}^{n} p_i \; ln\frac{p_i}{q_i}$$

Remark: $0 \ \log\frac{0}{q} = 0$, $p \ \log\frac{p}{0} = \infty$ for any nonnegative p and q.

(a) Prove that $\mathcal{E}(P|P) = 0$

(b) Prove that $\mathcal{E}(P|Q) \geq 0$

(c) Prove that $\mathcal{E}(P|Q) = 0$ if and only if $P = Q$ $(p_i = q_i, i = 1, \dots, n)$

(d) Provide a counterexample to demonstrate that $\mathcal{E}(Q|P) \neq \mathcal{E}(P|Q)$

5.(18 pts) Assume that the real-valued function $f(x)$, is countinuous and bounded, and $0 \leq t \leq T$, $-\infty < x < \infty$.

(a) Check that $u(t,x) = f(x - kt)$ is a solution to

$$\frac{\partial u}{\partial t}(t,x) + k\frac{\partial u}{\partial x}(t,x) = 0,$$

where k is any constant.

(b) Check that $u(t,x) = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi(T-t)}} e^{\frac{-x^2}{2(T-t)}} dx$ is a solution to

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) = 0 \\ u(t,x) = f(x). \end{cases}$$