國立清華大學 106 學年度碩士班考試入學試題

系所班組別：計量財務金融學系碩士班 乙組

考試科目（代碼）：微積分 (4604)

 *請在【答案卷、卡】作答

(15%) Part I: Multiple Choice.

1. Given a sequence $a_1 = \sqrt{2}, a_{k+1} = \sqrt{2 + a_k}, k \in N, \lim_{k\to\infty} a_k = ?$

   (a) $\sqrt{3}$ (b) 2 (c) 3

2. Let $f(x) : R \to R$ be continuous and satisfy $f(f(x)) = x, \forall x \in R$, can this equation $f(x) = x$ solvable?

   (a) Yes. (b) No. (c) Both are possible.

3. Let $f(x) = x^p, x \in [0,\infty)$. When $p \geq 1, f(x)$ is a ______ function.

   (a) convex (b) concave (c) both are possible.

4. Let $\alpha \in R$, $\lim_{x\to\infty} \frac{x^\alpha}{e^x} = ?$

   (a) 0 (b) 1 (c) ∞

5. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} = ?$

   (a) 0 (b) 1 (c) ∞

(18%) Part II: Fill in blanks.

1. If $0 \le a \le b \le c$, $\lim_{n\to\infty}\left(a^n + b^n + c^n\right)^{1/n} =$ ___(A)___ .

2. Let $\Gamma(\alpha) = \int_0^{\infty} e^{-x}x^{\alpha-1}dx, \alpha \in (0, +\infty)$, then

(1) for $\alpha > 0, \Gamma(\alpha+1) =$ ___(B)___.

(2) for $n \in N, \Gamma(n) =$ ___(C)___.

3. Calculate $\int_0^1 \left(\int_x^1 \tan^{-1} y\, dy\right) dx =$ ___(D)___ 。

4. The solution of $x\frac{dy}{dx} - 3y = x^2, x > 0$ is ___(E)___.

5. The solution of $\frac{dy}{dx} = \kappa y\left(\frac{m-y}{m}\right)$, where κ and m are constants, is ___(F)___.

(67%) Part II: Calculation and proof.

1. (16 %)

(1) State and derive Newton's method for solving an one-dimensional equation $f(x) = 0$.

(2) Briefly comments on its convergence rate.

2. (16%)

(1) Let $f(x) : [a, b] \to R$ be a C^1 function. Derive the formula to calculate the length of its graph.

(2) Calculate the length of $y = x^{3/2}, 0 \le x \le 2$.

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3. (24%) Let $f_\mu(x)$ denote the "exponential twist" of a real-valued density function $f(x)$, that means (1) $f(x) \geq 0$ for all $x \in R$ and (2) $\int f(x)dx = 1$, with parameter μ by

$$f_\mu(x) = \frac{e^{\mu x} f(x)}{M(\mu)},$$

where $M(\mu) = \int e^{\mu x} f(x)\, dx$.

(1) Derive the following upper bound when μ and c are positive

$$\int \mathbf{I}(x > c) \frac{f(x)}{f_\mu(x)} f(x) dx \leq M(\mu)\, e^{-\mu c}.$$

(2) Show that

$$\int x f_\mu(x) dx = \frac{M'(\mu)}{M(\mu)}.$$

(3) Suppose $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, then calculate $f_{\mu^\star}(x)$ when the parameter $\mu^\star$ is a solution of

$$\frac{M'(\mu^\star)}{M(\mu^\star)} = c.$$

4. (11%) Assume that the real-valued function $f(y)$ is differentiable for the one-dimensional variable $y > 0$ and $x \geq 0$ is a constant. Show that

$$f(y) = f(x) + \frac{df(x)}{dx}(y - x) + \int_x^\infty \frac{d^2 f(k)}{dk^2}(y - k)^+ dk + \int_0^x \frac{d^2 f}{dk^2}(k - y)^+ dk,$$

where $(z)^+ = \max\{z, 0\}$ denotes a maximum function.