國立清華大學 104 學年度碩士班考試入學試題

系所班組別：計量財務金融學系碩士班 乙組(財務工程組)

考試科目（代碼）：微積分(4604)

 *請在【答案卷、卡】作答

1. (8 pts) Let $S_{m.n} = \frac{m}{m+n}$. Find $\lim\limits_{n\to\infty}\left(\lim\limits_{m\to\infty} S_{m.n}\right)$ and $\lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty} S_{m.n}\right)$.

2. (12 pts) The Fibonacci sequence is defined by the equations

$$a_1 = a_2 = 1, \quad a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3.$$

Show that

$$\frac{1}{a_{n-1}a_{n+1}} = \frac{1}{a_{n-1}a_n} - \frac{1}{a_n a_{n+1}}$$

and deduce that

$$\sum_{n=2}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = 1, \qquad \sum_{n=2}^{\infty} \frac{a_n}{a_{n-1}a_{n+1}} = 2.$$

3. (12 pts) If $p > 0$, show that

$$\int_0^1 t^p (\ln t)^2 \, dt = \frac{2}{(p+1)^3}.$$

4. (12 pts) Prove that the equation

$$\cos x + 2\cos 2x + \cdots + n\cos nx = 0$$

has at least one real root between 0 and π.

5. (12 pts) Solve the initial value problem:

$$\frac{dy}{dx} = 2^{x-2y}, \qquad y(0) = \frac{1}{2}.$$

6. (12 pts) Evaluate the integral

$$\int\int_R e^{(x+y)/(x-y)} dx dy,$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

7. (14 pts) Let n be a positive integer.

   (a) Prove that

$$\left(1 - x^2\right)^n \geq 1 - nx^2 \quad \text{for all } x \in [0,1].$$

   (b) Prove that

$$\int_{-1}^1 \left(1 - x^2\right)^n dx \geq \frac{4}{3\sqrt{n}}.$$

8. (18 pts) Let p and q be positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

   (a) Show that the minimum of

$$f(x,y) = \frac{x^p}{p} + \frac{y^q}{q} \qquad (x > 0,\ y > 0)$$

   subject to the constraint $xy = 1$ is equal to 1.

(b) Use part (a) to show that if a and b are positive numbers, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(c) Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \cdots, n$, be positive numbers. Prove Hölder's inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$