一、微積分（共五十分）

1.（五分）請說明下列兩函數是否在其定義域中爲連續函數。

$$f(x)=\begin{cases}1 & if \quad x\ is\ rational\\ -1 & if \quad x\ is\ irrational\end{cases} \quad and \quad g(x)=|f(x)|$$

2.（五分）求出下列的極限值並詳細說明求解的過程。

$$\lim_{h\to 0}\frac{\sin^2 h}{h} \quad (\text{known } \lim_{h\to 0}\frac{\sin h}{h}=1)$$

3.（十分） (1) 若 $f(x)=x^2$ 及 $g(x)=|x|$ 證明 $f(g(x))$ 及 $g(f(x))$ 在 x=0 時可微分。
(2) 這與連鎖法則 (chain rule) 相矛盾嗎？

4.（十分）求出能使得函數

$$f(x)=x^2+\frac{a}{x}$$

在 (1)(2.5 分) $x=2$ 時有相對極小值的 a 值。
(2)(2.5 分) $x=-3$ 時有相對極小值的 a 值。
(3)(2.5 分) 反曲點在 $x=1$ 時的 a 值。
(4)(2.5 分) 證明此函數無相對極大值。

5.（五分）求下列極限值。

$$\lim_{x\to\frac{\pi}{2}}\frac{2x-\pi}{\cos x}$$

6.(五分）設函數 f 與 g 皆爲連續函數，且

$$\int_1^2 f(x)dx=-4 \ , \ \int_2^5 f(x)dx=6 \ , \ \int_1^5 g(x)=8$$

求出下式的值並說明過程。

$$\int_1^5[4f(x)-2g(x)]dx$$

7.（十分）解出下列兩積分值（詳列算式）。

(a) $\int 2x\sqrt{x^2-1}\,dx$ (b) $\int\frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$

二、統計（共五十分）

1. (10 points) Suppose $Cov(Y_1, Y_2) = 10$, $a = 2$, $b = 1$, $c = 1$, $d = 5$. Find $Cov(aY_1 + b, cY_2 + d)$

2. (8 points) Suppose $E(Y \mid X, Z) = aX + bZ$, $E(Z \mid X) = cX$. Find $E(Y \mid X)$.

3. (10 points) Let Y_1 and Y_2 denote the proportions of time, out of one workday, that employees I and II, respectively, actually spend on performing their assigned tasks. The joint relative frequency behavior of Y_1 and Y_2 is modeled by the density function

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \le y_1 \le 1;\ 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(1) Find $P(Y_1 < 1/2, Y_2 > 1/4)$.
(2) Find $P(Y_1 + Y_2 \le 1)$.

4. (10 points) Two brands of refrigerators, denoted by *A* and *B*, are each guaranteed for one year. In a random sample of 50 refrigerators of brand *A*, 12 were observed to fail before the guarantee period ended. A random sample of 60 brand *B* refrigerators also revealed 12 failures during the guarantee period. Estimate the true difference between proportions of failures during the guarantee period, $(p_1 - p_2)$, with confidence coefficient 0.95.

5. (12 points) Let $Y_1, \ldots Y_n$ denote a random sample from a Normal distribution with mean μ and variance σ^2. Let

$$\hat{\sigma}_1^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{1}{2}(Y_1 - Y_2)^2$$

(1) Show that both $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are unbiased estimators of σ^2.
(2) Find the efficiency of $\hat{\sigma}_1^2$ relative to $\hat{\sigma}_2^2$.

Appendix Tables

TABLE 3 Cumulative Normal Distribution $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998