

注意：考試開始鈴響前，不得翻閱試題，  
並不得書寫、畫記、作答。


國立清華大學 110 學年度碩士班考試入學試題

系所班組別：經濟學系

科目代碼：4503

考試科目：微積分與統計

### — 作答注意事項 —

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

# 國立清華大學 110 學年度碩士班考試入學試題

系所班組別：經濟學系碩士班 (0545)

考試科目 (代碼)：微積分與統計 (4503)

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\*請在【答案卷、卡】作答

## 微積分

- Please write in Chinese or English only.
  - Please make sure all your answers are legible, unquestionably labeled, and clearly explained (with equations if possible.)
  - For the safety of your own and others, please wear a mask.
1. The majority of the study of economics in general consists of solving constrained optimization problems. In microeconomics, usually a consumer is assumed to be with a set of preferences which can be represented by a *utility* function. This utility function “measures” how happy she is at any level of consumption of the goods that provide her with satisfaction.

- (a) (6 pts) Crystal lives only with two goods, coffee ( $x_1$ ) and chocolate ( $x_2$ ). Suppose that Crystal has the following quasi-linear utility function:

$$u(x_1, x_2) = \alpha_1 \ln x_1 + \alpha_2 x_2.$$

Each day, Crystal spends  $w$  dollars on coffee and chocolate. Suppose that the prices of coffee and chocolate are respectively  $p_1$  and  $p_2$ . That is, the budget constraint of Crystal can be written as:

$$p_1 x_1 + p_2 x_2 = w.$$

Suppose that there is an inner solution. Derive  $x_1(p_1, p_2, w)$  and  $x_2(p_1, p_2, w)$ , the demand functions of coffee and chocolate of Crystal. That is, solve the following problem:

$$\begin{aligned} \max_{(x_1, x_2) \in \mathbb{R}^2} & \alpha_1 \ln x_1 + \alpha_2 x_2, \\ \text{such that} & p_1 x_1 + p_2 x_2 = w, \end{aligned}$$

and write  $x_1$  and  $x_2$  as functions of  $p_1$ ,  $p_2$ , and  $I$ .

- (b) (3 pts)  $v(p_1, p_2, w)$ , the indirect utility function, gives Crystal's maximal attainable utility when faced with  $(p_1, p_2)$ , the vector of goods prices, and  $w$ , the amount of income:

$$v(p_1, p_2, w) = u[x_1(p_1, p_2, w), x_2(p_1, p_2, w)].$$

Write down the indirect utility function of Crystal. That is, plug in your answers of (a) into the quasi-linear utility function.

- (c) (6 pts) *Roy's identity* is a major result in microeconomics having applications in consumer choice and the theory of the firm. The lemma relates the demand function to the derivatives of the indirect utility function. Specifically, demand function for good  $i$  can be calculated as

$$x_i = -\frac{\partial v / \partial p_i}{\partial v / \partial w}.$$

Calculate  $\partial v(p_1, p_2, w) / \partial p_1$ ,  $\partial v(p_1, p_2, w) / \partial p_2$ , and  $\partial v(p_1, p_2, w) / \partial w$ .

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2. In macroeconomics, the Ramsey-Cass-Koopmans model assumes that the household's *lifetime* utility function takes this form:

$$U = \int_{t=0}^{\infty} \exp(-\rho t) u(C(t)) dt,$$

where  $C(t)$  is the *instantaneous* consumption of the household at time  $t$ ,  $u(\cdot)$  is the *instantaneous* utility function, and  $\rho$  is the discount rate.  $0 < \rho < 1$ . Usually the instantaneous utility function is assumed to take the *constant-relative-risk-aversion* (CRRRA) form:

$$u(C(t)) = \frac{C(t)^{1-\theta} - 1}{1-\theta},$$

where  $\theta > 0$ .

- (a) (5 pts) Suppose that Loretta's instantaneous utility function takes the CRRRA form. Compute her coefficient of relative risk aversion:

$$-\frac{Cu''(C)}{u'(C)},$$

where  $u'(C)$  and  $u''(C)$  are respectively the first and second derivatives of  $u(C)$ .

- (b) (5 pts) Following (a), suppose that Loretta's instantaneous consumption is at the steady state. That is,  $C(t) = \bar{C}$ . Determine Loretta's lifetime utility:

$$U = \int_{t=0}^{\infty} \exp(-\rho t) u(\bar{C}) dt.$$

- (c) (5 pts) Following (a), suppose that Loretta's instantaneous utility function takes the CRRRA form with  $\theta = 1$ . Determine the instantaneous utility of Loretta. That is, determine

$$\lim_{\theta \rightarrow 1} \frac{C(t)^{1-\theta} - 1}{1-\theta}.$$

3. In capital theory, one principle for the determining of the correct price of the firm's capital is given by the equation:

$$p(t)K(t) = \int_t^{\infty} R(\tau)K(\tau) \exp(-r(\tau-t)) d\tau,$$

where  $K(t)$  denotes the capital stock of some firm at time  $t$ ,  $p(t)$  denotes the price per unit of capital,  $R(t)$  denotes the rental price per unit of capital, and  $r$  is the constant interest rate. That is, the current cost of capital should equal the discounted present value of the returns from lending it.

- (a) (5 pts) Eric would like to find an expression for  $R(t)$  by differentiating the equation with respect to  $t$ . Take the derivative of the left hand side:

$$\frac{d}{dt} \int_t^{\infty} R(\tau)K(\tau) \exp(-r(\tau-t)) d\tau.$$

- (b) (5 pts) Following (a), take the derivative of the right hand side and find the expression for  $R(t)$ .

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4. (5 pts) In the Dixit-Stiglitz model, the representative consumer's utility function can be written as:

$$U = \left( \int_0^n q(\omega)^\rho d\omega \right)^{1/\rho},$$

where  $q(\omega)$  is consumption of variety  $\omega$ ,  $n$  is the mass of varieties available to consumers, and  $\rho$  is a measure of substitutability.  $0 < \rho < 1$ .

Suppose that for any  $\omega \in (0, 1)$ , Stephen consumes  $q(\omega) = \exp(\omega)$ . Determine Stephen's utility. That is, determine

$$\left( \int_0^n (\exp(\omega))^\rho d\omega \right)^{1/\rho}.$$

5. (5 pts) In Econometrics, nowadays ridge, lasso, and elastic net are widely used. In general, for the following estimating equation:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_p x_{pi} + e_i,$$

an elastic net regression can be written as:

$$\min_{\beta_0, \beta_1, \beta_2, \dots, \beta_p} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \cdots - \beta_p x_{pi})^2 - \lambda_1 \sum_{j=1}^p |\beta_j| - \lambda_2 \sum_{j=1}^p \beta_j^2,$$

where  $\lambda_1 \geq 0$  and  $\lambda_2 \geq 0$ . When  $\lambda_1 = 0$ , the elastic net reduces to a ridge; when  $\lambda_2 = 0$ , the elastic net reduces to a lasso.

For simplicity, Ray consider a one-dimensional ridge. That is, for  $\lambda > 0$ , let

$$\hat{\beta}_{\text{Ridge}} = \arg \min_{\beta \in \mathbb{R}} \sum_{i=1}^n (y_i - \beta x_i)^2 - \lambda \beta^2.$$

Determine  $\hat{\beta}_{\text{Ridge}}$ , the ridge estimator for  $\beta$ .



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## 統計

1. A random variable;  $X$ , has a probability density function that can be specified as:

$$f(x) = \begin{cases} \theta x(1-x), & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

- (a) (3 points) Find the value of  $\theta$ .
- (b) (6 points) Based on your answer in part (a), calculate the expected value of  $X$ , denoted as  $E(X)$ , and variance, denoted as  $\text{Var}(X)$ .
- (c) (3 points) Find  $\text{Pr}(X > 0.4)$
2. Suppose that  $X_1, X_2, \dots, X_n$  are  $n$  random variables that are drawn independently from a population that is normally distributed with a mean  $\mu$  and variance  $\sigma^2$ .  $n$  is an even number. A researcher is considering three alternative estimators of the population mean  $\mu$ . The first estimator is:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

The second estimator is:

$$\tilde{X} = \frac{2}{n}(X_1 + X_3 + \dots + X_{n-1}).$$

here, we assume that  $n$  is an even number and only the odd numbered  $X$ 's are included in the estimator. For instance, if  $n = 10$  and the estimator is:  $\tilde{X} = \frac{2}{n}(X_1 + X_3 + X_5 + X_7 + X_9)$ . The third estimator is:

$$\hat{X} = X_1$$

- (a) (4 points) Are these three estimators unbiased? Explain and show your work.
- (b) (4 points) Which one is relatively more efficient? Explain and show your work.
- (c) (4 points) Are these three estimators consistent? Explain and show your work.
3. A researcher is considering a simple linear model without a constant term:

$$y_i = \beta_1 x_i + \epsilon_i,$$

where  $\epsilon_i$  is a random error term,  $\epsilon \sim N(0, \sigma^2)$  and  $\text{cov}(x, \epsilon) = 0$ . The equation estimated is given by:

$$\tilde{y}_i = \tilde{\beta}_1 x_i,$$

where  $\tilde{\beta}_1$  is the ordinary least squares (OLS) estimator of  $\beta_1$  and  $\tilde{y}$  is the fitted value of  $y$ .

- (a) (5 points) Derive the OLS estimator  $\tilde{\beta}_1$ .
- (b) (5 points) Calculate the variance of  $\tilde{\beta}_1$ .

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- (c) (5 points) If the true data generating process contains a constant term such that

$$y_i = \beta_0 + \beta_1 x_i + v_i.$$

where  $v_i$  is a random error term. Is the estimator derived in part (a) unbiased? Show your work and explain carefully.

4. A researcher is investigating determinants of academic performance among high school students. Suppose that academic performance (measured by GPA) depends on gender (male or female) and whether the student lives in urban areas (urban or rural). The model can be specified as:

$$GPA_i = \beta_0 + \beta_1 female_i + \beta_2 urban_i + \beta_3 female_i \times urban_i + \epsilon_i$$

where  $female_i$  is a dummy variable that indicates if the student is female, that is,  $female_i = 1$  if student  $i$  is female and  $female_i = 0$  if student  $i$  is not female (meaning that student  $i$  is male).  $urban_i$  is a dummy variable indicating if student  $i$  lives in urban areas, that is,  $urban_i = 1$  if student  $i$  lives in urban areas; and  $urban_i = 0$  if student  $i$  does not live in urban areas (meaning that the student lives in rural areas).  $female_i \times urban_i$  is an interaction term of these two variables.  $\epsilon_i$  is a random error term.

- (a) (3 points) Another researcher asks why this equation does not include both a female indicator (female = 1 if female and = 0 if not), and a male indicator (male = 1 if male and = 0 if not). Explain why or why not.
- (b) (4 points) What is the expected GPA of male students living in rural areas? What is the expected GPA of female students living in rural areas? Express in terms of  $\beta$ 's and show your work.
- (c) (4 points) What is the difference between the expected GPA of female students living in urban areas and the expected GPA of male students living in urban areas? What is the difference between male students living in urban areas and male students living in rural areas? Express in terms of  $\beta$ 's and show your work.