國立清華大學 107 學年度碩士班考試入學試題

系所班組別: 經濟學系(0545)
考試科目(代碼): 微積分與統計(4503)

 *請在【答案卷、卡】作答

1. (10 points) Evaluate the following limits:

   (a)
$$\lim_{x\to\infty} \frac{e^x + 4x}{e^{2x} + x^2}.$$

   (b)
$$\lim_{x\to\infty} \sqrt[x]{x}.$$

2. (10 points) Integrate
$$\int x^2 e^{-x} dx.$$

3. (10 points) Find the Taylor series for
$$f(x) = x^3 - 8x^2 + 6$$
   about $x = 3$.

4. (10 points) Examine the function
$$f(x) = (8 - x)^4$$
   for its relative extremum.

5. (10 points) Find the extremum of
$$f(x, y) = x^2 y^3$$
   subject to
$$2x + y = 5.$$

6. (3 points) Suppose you're on a game show *Let's Make a Deal* hosted by Monty Hall, and you're given the choice of 4 doors: Behind one door is a Maserati; behind the other 3, goats. You pick a door, and Monty, who knows what's behind the doors, opens 2 doors among the 3 unpicked ones, which have goats. He then says to you, "Do you want to switch?" What is the probability for you to win the Maserati if you switch?

7. (3 points) The world-famous National Tsing Hua University and another not-so-famous school are going to have the annual Plum-Bamboo Tournament on March 2, 2018. In recent 20 years, in has rained only once on that day:

$$P\left(\text{rain on March 2}\right) = 0.05.$$

Unfortunately, the weatherman has predicted rain for March 2. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time:

$$\begin{aligned} P\left(\text{weatherman forecasts rain}|\text{rain on March 2}\right) &= 0.9, \\ P\left(\text{weatherman forecasts rain}|\text{no rain on March 2}\right) &= 0.1. \end{aligned}$$

What is the probability that it rains on the Plum-Bamboo Tournament given the forecast?

$$P\left(\text{rain on March 2}|\text{weatherman forecasts rain}\right) =?$$

8. (3 points) A baseball player will be at bats for a number of times in a season and has a batting average $p$. Let $X =$ "number of hits" and $Y =$ "number of at bats". Suppose

$$\begin{aligned} Y &\sim \text{Poisson}\left(\lambda\right), \\ X|Y &\sim \text{binomial}\left(Y, p\right). \end{aligned}$$

What is the marginal distribution of $X$, the number of hits?

9. Suppose $X_1, \ldots, X_n$ are independent and identically normal-distributed with a mean $\mu_0$ and a variance $\sigma_0^2$, *i.e.*, $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathrm{n}(\mu_0, \sigma_0^2)$. Let

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2, \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2.$$

   (a) (2 points) Is $\overline{X}$ an unbiased estimator of $\mu_0$?
   (b) (2 points) What is the variance of $\overline{X}$?
   (c) (2 points) Is $\widehat{\sigma}^2$ an unbiased estimator of $\sigma_0^2$?
   (d) (2 points) Is $S^2$ an unbiased estimator of $\sigma_0^2$?
   (e) (2 points) Is $\overline{X}$ the best unbiased estimator of $\mu_0$? That is, is $\overline{X}$ the uniform minimum variance unbiased estimator (UMVUE) of $\mu_0$? Why?
   (f) (2 points) Is $\overline{X}$ a consistent estimator of $\mu_0$?
   (g) (2 points) Is $\widehat{\sigma}^2$ a consistent estimator of $\sigma_0^2$?
   (h) (2 points) Is $S^2$ a consistent estimator of $\sigma_0^2$?
   (i) (2 points) What is the sampling distribution of $\overline{X}$?
   (j) (2 points) What is the sampling distribution of $(n-1)\,S^2/\,\sigma^2$?
   (k) (2 points) What is the sampling distribution of $\left(\overline{X} - \mu_0\right)/(S\,/\sqrt{n})$?
   (l) (2 points) When $n \to \infty$, what is the limiting distribution of $\left(\overline{X} - \mu_0\right)/(\sigma_0\,/\sqrt{n})$?
   (m) (2 points) When $n \to \infty$, what is the limiting distribution of $\left(\overline{X} - \mu_0\right)/(S\,/\sqrt{n})$?

10. Willy Wonka, the owner of the chocolate factory recently designs a new way to inspect the quality of his products, the Wonka bars. When inspecting a Wonka bar, Willy randomly grabs a ball from an urn with 19 white balls and 1 black ball. The Wonka bar inspected is qualified if a white ball is grabbed out, and the Wonka bar inspected is not qualified if the black ball is grabbed out. Consider the null hypothesis $H_0$ : the Wonka bar inspected is qualified.

   (a) (2 points) What is the type I error of Willy's test?
   (b) (2 points) What is the type II error of Willy's test?
   (c) (2 points) What is the power of Willy's test?

11. Consider the following linear regression model:

$$Y_i = \alpha + \beta X_i + U_i, \quad i = 1, \ldots, n,$$

(a) (3 points) What is the ordinary least squares estimator of $\beta$?

(b) (3 points) Let $\widehat{\beta}$ denote the ordinary least squares estimator of $\beta$. Suppose the true data generating process is

$$Y_i = bX_i + V_i,$$

where $\{X_i\}$ is a sequence of non-random variables, and $\{V_i\}$ is a sequence of independent and identically distributed random variables with $\mathrm{E}(V_i) = 0$ and $\mathrm{Var}(V_i) = \sigma_v^2$. What is the expected value of $\widehat{\beta}$?

(c) (3 points) Again, let $\widehat{\beta}$ denote the ordinary least squares estimator of $\beta$. Suppose the true data generating process is

$$Y_i = bX_i + cZ_i + W_i,$$

where $\{X_i\}$ is a sequence of non-random variables, $\{Z_i\}$ is a sequence of independent and identically distributed random variables with $\mathrm{E}(Z_i) = a$ and $\mathrm{Var}(Z_i) = \sigma_z^2$, and $\{W_i\}$ is a sequence of independent and identically distributed random variables with $\mathrm{E}(W_i) = 0$ and $\mathrm{Var}(W_i) = \sigma_w^2$. What is the expected value of $\widehat{\beta}$?