1 (18%)

Which of the following are analytic functions in z $(z = x + iy)$?

Give your reasons briefly.

(a) $f_a(z) = z^*$;

(b) $f_b(z) = \operatorname{Im} z$;

(c) $f_c(z) = z^2$;

(d) $f_d(z) = e^{z^2}$;

(e) $f_e(z) = e^{|z|}$;

(f) $f_f(z) = 10$.

2 (12%)

A matrix M is

$$M = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix}.$$

(a) Find the eigenvalues of M.

(b) Find the normalized eigenvectors of M.

(c) Are the eigenvectors orthogonal? Comment on this.

3 (30%)

Evaluate the integrals

(a) $I_a = \int_0^{2\pi} \frac{4\,dx}{5 - 4\cos x}$;

(b) $I_b = \int_0^{\infty} \frac{\sin kx}{x}\,dx$; and

(c) $I_c = \int\int \vec{r} \cdot d\vec{\sigma}$ over the surface of a sphere of radius R.

4 (15%)

Find the Fourier Expansion of the function

$$f(x) = \begin{cases} 0 & 0 \le x \le a \\ 2 & a \le x \le b \end{cases}.$$

5 (15%)

Use $e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$ to find the expressions of $J_n(x)$, when $n \ge 0$ and $n < 0$.

6 (10%)

For the Helmholtz equation

$$(\nabla^2 + k^2)U(\vec{r}) = -4\pi\delta(\vec{r}),$$

find a solution that is spherically symmetric and goes to zero at infinity.