類組：物理類　科目：應用數學(2001)　　　　　　　　　　共 2 頁 第 1 頁

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參考用

計算題，請詳列計算過程，無計算過程不予計分。

1. Solve the following first-order ordinary differential equation (ODE, 15 points)

   (a) $(1-x^2)y' = xy + 2x\sqrt{1-x^2}$

   (b) $y' + y = xy^{2/3}$

   (c) $y' = xy^2 - \frac{2}{x}y - \frac{1}{x^3}$

2. (ODE, 10 points) Solve $y'' - 8y' + 16y = 32t$ with $y(0) = 1$ and $y'(0) = 0$. *Hint*: You may want to use the Laplace transformation.

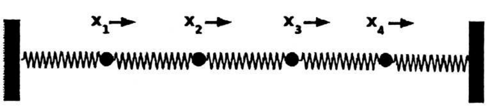
3. (Second-order ODE and SHM, 20 points)

   (a) Solve the damped simple harmonic motion, $m\ddot{x} = -kx - \alpha\dot{x}$ with $\alpha > 0$, for the under- and critically-damped cases, respectively.

   (b) With the addition of an external drive, $m\ddot{x} = -kx - \alpha\dot{x} + F_0\sin\Omega t$. Find the steady-state solution for the under-damped case.

4. (Eigenvalue problems, 10 points)

   Find the characteristic frequencies and modes for a series of springs. Denote the particle mass by m and spring constant k.



5. (Random walk, 15 points)

   (a) (5 points) Derive Stirling's formula: $\lim_{n\gg1} n! \sim n^n e^{-n}\sqrt{2\pi n}$.

   (b) (10 points) Given the binomial probability of finding a drunkard who left pub at n=0 and ends up at $n \neq 1$ after N steps equals $P(n,N) = \frac{N!}{\left(\frac{N+n}{2}\right)!\left(\frac{N-n}{2}\right)!}\frac{1}{2^N}$.

   Show that it can be reduced to Gaussian distribution $\frac{1}{\sqrt{2\pi N}}e^{-\frac{n^2}{2N}}$ if $N \gg n \gg 1$.

6. (Laplace transform, gamma function, and fractional calculus, 15 points)

   Studying acoustic waves in biological tissue requires fractional differentiation.

   (a) Find the Laplace transformation of x^n, i.e., $\int_0^\infty x^n e^{-sx}dx$.

   (b) If $\frac{d^m}{dx^m}e^{ax} = a^m e^{ax}$ can be generalized to non-integer m, find $\frac{d^{0.5}}{dx^{0.5}}x^n$.

   (c) Please generalize to solve $\frac{d^{0.5}}{dx^{0.5}}\sin(ax)$ and $\frac{d^{0.5}}{dx^{0.5}}\tan(ax)$.

注意：背面有試題

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7. (Contour integral and Cauchy's theorem, Fourier series, 15 points)

Integrate the infinite geometric series $\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots$ gives $x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$.

Divide by x and integrate again then produces $x + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \cdots$. As a result, we can obtain

$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{1-y^2} = \int_0^1 \frac{dx}{2x} ln \frac{1+x}{1-x}$.

(a) (5 points) Set $\frac{1+x}{1-x} = e^z$ and show the right-hand-side becomes $\int_0^\infty \frac{z}{e^z - e^{-z}} dz$. Solve it by doing a seemingly irrelevant integral, $\oint \frac{z^2}{e^z - e^{-z}} dz$, along the close contour that connects $\infty + i\pi \to -\infty + i\pi$ and $-\infty - i\pi \to \infty - i\pi$ at $Re\, z = \pm\infty$.

(b) (5 points) If you cannot solve (a), it is okay to denote $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ by "A". Express $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ and $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ in terms of A.

(c) (5 points) A more handsome trick to solve $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ is credited to Euler who argued that, since $\frac{\sin x}{x}$ is even in x and equals 1 at x=0,

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots. \quad (1)$$

(i) Taylor-expand Eq.(1) to $O(x^2)$ to find $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.

(ii) Replace the x in Eq.(1) by ix and multiply it by Eq.(1). Now the first order in Taylor-expansion becomes $O(x^4)$. Find $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$.