科目：高等微積分(1001)　校系所組：中大數學系甲組
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1. (12%) If g is continuous and nonnegative on $[0,1]$ with $g(0) = 0 = g(1)$, show that there exists $a, b \in [0,1]$ such that $g(a) = g(b)$ and $b - a = 1/2$.

2. (12%) If for each n, $|f_n(x)| \leq M_n$, $x \in E \subset \mathbb{R}$, and if f_n converges uniformly on E to f. Show that there is $M > 0$ such that $|f_n(x)| \leq M$ for all n and $x \in E$.

3. (12%) Let $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$, is f differentiable at $(0,0)$? Give your reason.

4. (12%) Find the extrema of $f(x,y,z) = x^4 + y^4 + z^4$, on the circle defined by
$$x^2 + y^2 + z^2 = 1 \text{ , and } x + y + z = 1.$$

5. (15%) Consider the space $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ and for $f,\ g \in C([0,1])$, define
$$d(f,g) = \int_0^1 |f(x) - g(x)| dx.$$
   (a) (5%) Show that d is a metric on $C([0,1])$.
   (b) (5%) Is the metric space $(C([0,1]), d)$ complete? Explain!
   (c) (5%) Is the set $\{f \in C([0,1]) : f(x) > 0, \text{ for all } x \in [0,1]\}$ open in the space $(C([0,1]), d)$? Explain!

6. (15%) Let
$$F(x) = \int_1^\infty t^{x-1} e^{-t} dt,\ x \in \mathbb{R}.$$
   (a) (7%) Show that $F(x)$ exists for $x \in \mathbb{R}$.
   (b) (8%) Show that F is differentiable for $x \in \mathbb{R}$ and find $F'(x)$.

7. (10%) Is there a continuous function from $[0,1]$ onto $(0,1)$? Prove your assertion.

8. (12%) Let $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ denote the unit half-sphere in $\mathbb{R}^3$. Evaluate the surface integral over S:
$$\int_S (x + y + z^2) dS,$$
where dS is the area element on S.