科目：代　　數(1004)　　校系所組：清大數學系純粹數學組

## ALGEBRA

(20%) 1. Let $Q$ be the field of rational numbers.

(10%) (a) Prove that the polynomial $f(x) = x^3 + 3x^2 - 9x + 6 \in Q[x]$ is irreducible over $Q$.

(10%) (b) Let $Q(u)$ be the simple extension field of $Q$ generated by a root $u \in \mathbb{C}$ of the polynomial $f(x)$ in (a). Express $(u+1)^{-1} \in Q(u)$ in terms of the $Q$-basis $\{1, u, u^2\}$ for $Q(u)$.

(15%) 2. Let $H$ be a cyclic group. Prove that if $H$ is a normal subgroup of some group $G$ then any subgroup $K$ of $H$ is also normal in $G$.

(25%) 3. Given a group $G$. Let $C(G)$ be the subgroup of $G$ generated by the set of all elements of the form $aba^{-1}b^{-1}$, $a, b \in G$. $C(G)$ is a normal subgroup of $G$ (you don't have to prove this).

(7%) (a) Prove that the quotient group $G/C(G)$ is an abelian group.

(8%) (b) Let $G \xrightarrow{\phi} G'$ be a group homomorphism from $G$ to an abelian group $G'$. Prove that $C(G) \subseteq \ker\phi$.

(10%) (c) Let $S_3$ be the symmetric group on 3 letters. Find $C(S_3)$.
(**Hint** : Recall that there is a group homomorphism $S_3 \xrightarrow{\phi} \mathbb{Z}_2$ which is onto.)

(20%) 4. Consider the ring of Gaussian integers $\mathbb{Z}[i] = \{x + yi \mid x, y \in \mathbb{Z}, i = \sqrt{-1}\}$.

(10%) (a) For a non-zero principal ideal $J =< a + bi >$ in $\mathbb{Z}[i]$, prove that the quotient ring $\mathbb{Z}[i]/J$ is a finite ring. (**Hint** : $\mathbb{Z}[i]$ is a Euclidean domain with norm $\nu(x + yi) = x^2 + y^2$.)

(10%) (b) What is the number of elements in $\mathbb{Z}[i]/J$ for $J =< 1 + i >$? Prove your answer.

(20%) 5. Let $K$ be a field. Let $K[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in K\}$ be the ring of formal power series over $K$.

(12%) (a) Prove that $a_0 + a_1x + a_2x^2 + \cdots$ is a unit in the ring $K[[x]]$ if $a_0 \neq 0$.

(8%) (b) Let $x = 0 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$ and let $(x)$ be the principal ideal of $K[[x]]$ generated by $x$. Prove that $(x)$ is the only maximal ideal of $K[[x]]$.