國 立 清 華 大 學 命 題 紙

95學年度 數學系(所) 應數組 碩士班入學考試

科目 線性代數 科目代碼 0202 共 1 頁第 1 頁 *請在【答案卷卡】內作答

1. (14%) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Compute $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

2. (14%) Consider all 3×3 matrices A satisfying

(a) The right (column) eigenvectors of A are $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) The eigenvalues of A are distinct.

Show that all such matrices A have the same left (row) eigenvectors (up to a constant multiplication).

3. (14%) Let $g(\mathbf{x}) = ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_2x_3 + fx_1x_3$. Show that there exists $\mathbf{y}, \mathbf{z} \in \mathbf{R}^3$ such that

(a) $\|\mathbf{y}\| = \|\mathbf{z}\| = 1$, .

(b) $g(\mathbf{y}) \le g(\mathbf{x}) \le g(\mathbf{z})$ for all $\mathbf{x} \in \mathbf{R}^3$ with $\|\mathbf{x}\| = 1$ and

(c) $\mathbf{y} \perp \mathbf{z}$.

4. Let $\mathcal{M}_2(\mathbf{R})$ be the vector space of all 2×2 real matrices and let $P = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$. Define $T : \mathcal{M}_2(\mathbf{R}) \to \mathcal{M}_2(\mathbf{R})$ such that $T(A) = PA$.

(a) (7%) Find the minimal polynomial of T.

(b) (7%) Find the Jordan canonical form of T.

5. (14%) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathbf{u} \in \mathbf{R}^2$ such that $\|A\mathbf{u} - \mathbf{y}\| \le \|A\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x} \in \mathbf{R}^2$. Is the $\mathbf{u}$ unique ? Give your reason.

6. Let V be a inner product space and let A_j be the orthogonal projections to the subspace W_j for $j = 1, 2$.

(a) (10%) Show that $A_1A_2 = A_2A_1$ if and only if $W_1 = (W_1 \cap W_2) \oplus (W_1 \cap W_2^{\perp})$.

(b) (6%) In case (a) is true, show that A_1A_2 is the orthogonal projection to $W_1 \cap W_2$.

7. A 3×3 real matrix T is said to be in $SO(\mathbf{R}, 3)$ if $\det(T) = 1$ and $\|T\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{R}^3$.

(a) (7%) Show that for such T there is a nontrivial $\mathbf{u} \in \mathbf{R}^3$ such that $T\mathbf{u} = \mathbf{u}$.

(b) (7%) Find the Jordan canonical form of such T.