國 立 清 華 大 學 命 題 紙

95學年度 數 學 系（所） 純 數 組 碩士班入學考試

科目 代數及線性代數 科目代碼 0102 共 2 頁第 1 頁 *請在【答案卷卡】內作答

1. Explanation is required for your examples.

(i) [5] Let A_5 denote the alternating group on 5 letters. Check that all Sylow subgroups of A_5 are abelian.

(ii) [5] Give an example of an ideal I of a commutative ring R such that I is prime but not maximal.

(iii) [5] Give an example of a unique factorization domain which is not a principle ideal domain.

2. A group G is called a *metabelian group* if G has a normal subgroup K such that both K and G/K are abelian.

(i) [5] Give an example of a non-abelian group G which is metabelian.

(ii) [10] Show that G is metabelian if and only if the commutator subgroup $[G, G]$ of G is abelian.

3. An element m in an integral domain D is called a *least common multiple* of elements $a_1, \ldots, a_n$ if the following two conditions are satisfied:

(1) m is a multiple of a_i for each i;

(2) if r is a multiple of each a_i, then r is a multiple of m.

Now fix a prime p in $\mathbf{Z}$ and let D denote the set of all polynomials in $\mathbf{Z}[x]$, with the coefficient of x divisible by p.

(i) [5] Show that D is an integral domain.

(ii) [10] Show that the least common multiple of p and px in D does not exist.

4. Let E be an extension field of a field F and let R be a subring of E such that $F \subseteq R \subseteq E$.

(i) [10] Suppose E is an algebraic extension of F. Prove that R is a field.

(ii) [5] Give an example such that R is not a field if E is not an algebraic extension of F.

5. Let $\mathcal{M}_2(\mathbf{R})$ be the vector space of all 2 by 2 real matrices and let $P = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$. Define $T : \mathcal{M}_2(\mathbf{R}) \to \mathcal{M}_2(\mathbf{R})$ such that $T(A) = PA$.

(a) [7] Find the minimal polynomial of T.

(b) [7] Find the Jordan canonical form of T.

6. [14] Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathbf{u} \in \mathbf{R}^2$ such that $\|A\mathbf{u} - \mathbf{y}\| \le \|A\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x} \in \mathbf{R}^2$. Is the $\mathbf{u}$ unique ? Give your reason.

7. Let V be a inner product space and let A_j be the orthogonal projections to the subspace W_j for $j = 1, 2$.
   (a) [10] Show that $A_1A_2 = A_2A_1$ if and only if $W_1 = (W_1 \cap W_2) \oplus (W_1 \cap W_2^{\perp})$.
   (b) [6] In case (a) is true, show that A_1A_2 is the orthogonal projection to $W_1 \cap W_2$.

8. A 3 by 3 real matrix T is said to be in $SO(\mathbf{R}, 3)$ if $\det(T) = 1$ and $\|T\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{R}^3$.
   (a) [8] Show that for such T there is a nontrivial $\mathbf{u} \in \mathbf{R}^3$ such that $T\mathbf{u} = \mathbf{u}$.
   (b) [8] Find the Jordan canonical form of such T.