國 立 清 華 大 學 命 題 紙

95學年度 數 學 系（所） 純 數 組 碩士班入學考試

科目 高等微積分 科目代碼 0101 共 1 頁第 1 頁 *請在【答案卷卡】內作答

There are 8 problems (15 points for each).

1. Check whether the following limits exist and find their values.

   (a) $\lim_{n\to\infty} \frac{\ln(n)}{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}$.

   (b) $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$.

2. Suppose $f \in C([a,b]) \cap C^1((a,b))$, $f(a) = f(b) = 0$ and $\lim_{x\to a+} f'(x) = \lim_{x\to b-} f'(x) > 0$, show that f has at least one zero in (a,b).

3. Let D be a region in $\mathbb{R}^2$ and $u(x,y) \in C^2(D)$. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on D if and only if for any ball $B \subset D$, $\int_{\partial B} (\nabla u \bullet n) ds = 0$, where n is the unit outward normal field to ∂B and ds is the arclength element.

4. (a) Suppose A, B are subsets of $\mathbb{R}^n$ and $f : A \to B$ is a homeomorphism. Show that f maps the interior of A onto the interior of B.

   (b) Use (a) to show that the closed annulus $A = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2\}$ is not homeomorphic to the closed disc $D = \{x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

5. Suppose $a_k \in \mathbb{R}$ and $|a_k| \le k$ for every positive integer k. Let $f(x) = \sum_{k=1}^{\infty} a_k x^k$ and $f_n(x) = f(x + \frac{1}{n})$. Show that f_n converges uniformly to f on any $[a,b] \subset (-1,1)$.

6. Define $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $f(A) = A^T A$, where A^T denotes the transpose of A. Use the *definition of derivative* to find $Df(I_n)$, where I_n is the identity matrix.

7. Let S be the set defined by

   $$S = \{(x,y,u,v) \in \mathbb{R}^4 : 3xy + y^3 + e^u + e^v = 0,\ x^3 - y^2 + e^u + e^{-v} = 0\}.$$

   Show that there is a neighborhood U of $(-1,1)$ and real-valued functions f, g such that $(x,y,f(x,y),g(x,y)) \in S$ for any $(x,y) \in U$. Find the value $\frac{\partial}{\partial x} f(-1,1)$.

8. Evaluate the double integral

   $$\iint_E e^{5x^2+2xy+y^2}\, dA$$

   where E is the ellipse $\{(x,y) \in \mathbb{R}^2 : 5x^2 + 2xy + y^2 \le 1\}$.