國立清華大學命題紙

94學年度 數 學 系(所) 應數 組碩士班入學考試

科目 線性代數 科目代碼 0202 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

Linear Algebra Examination

1. (25 points) Determine "true" or "false" for the following statements. Briefly sketch your proof when the answer is "true", explain why or give a counterexample when the answer is "false".

   (a) If $A \in \mathrm{M}_{m\times n}(\mathbb{R})$, $B \in \mathrm{M}_{n\times m}(\mathbb{R})$, and $m \neq n$, then AB is singular.

   (b) Let S, T be linear operators on an n-dimensional vector space such that $ST = 0$. Then $\mathrm{rank}(S) + \mathrm{rank}(T) \leq n$.

   (c) For any u_1, $u_2 \in \mathbb{R}^5$, v_1, $v_2 \in \mathbb{R}^4$, $u_1 \neq u_2$ and $v_1 \neq v_2$, there is a 4×5 matrix A such that $Au_1 = v_1$, $Au_2 = v_2$.

   (d) Any matrix of the form $\begin{pmatrix} 1 & a & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is diagonalizable.

   (e) Let B be obtained from $A \in \mathrm{M}_{5\times 5}(\mathbb{C})$ by moving the i-th row to the $(i+1)$-th row, $i = 1, \cdots, 4$, and moving the fifth row to the first row. Then $\det(A) = \det(B)$.

2. (15 points) Let $u = (2,1,0)$, $v = (3,0,2)$, $w = (0,-2,3)$. Suppose T is a linear operator on $\mathbb{R}^3$ that interchanges u and v, and maps w to $(1,0,0)$. Find the matrix representation $[T]_\beta$ of T with respect to the standard basis $\beta = \{e_1, e_2, e_3\}$.

3. (18 points) Let T be a linear operator on a finite dimensional vector space V. Suppose T is idempotent; that is, $T^2 = T$. Prove that

   (a) Eigenvalues of T are either 0 or 1.

   (b) $V = \ker(T) \oplus \mathrm{range}(T)$.

   (c) T is diagonalizable.

4. (12 points) Let T be a self-adjoint operator (that is, $T = T^*$) on $\mathbb{C}^n$.

   (a) Show that all eigenvalues of T are real.

   (b) Suppose λ, μ are distinct eigenvalues of T and E_λ, E_μ are the corresponding eigenspaces. Show that $E_\lambda \subset E_\mu^\perp$.

5. (15 points) On the space of continuous real-valued functions on $[0,1]$, define an inner product by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

   Let $f_1(x) = x$, $f_2(x) = 4x^2$. Find the orthogonal projection of $g(x) = 3 + 4x$ on the linear subspace generated by f_1, f_2.

6. (15 points) Let $P_3(\mathbb{C})$ be the space of complex polynomials of degree less than or equal to 3. Define the linear operator T on $P_3(\mathbb{C})$ by

$$T(f) = -\frac{x^2}{2}f''' + f'' + f' - 2f.$$

Find the Jordan canonical form and minimal polynomial of T.