國 立 清 華 大 學 命 題 紙

94 學年度 數 學 系（所） 應數 組碩士班入學考試

科目 高等微積分 科目代碼 0201 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

Show your work, otherwise no credit will be granted.

1. (15 points)
   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Is $f^{-1}(K)$ compact for every compact subset K of $\mathbb{R}$?
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If $f^{-1}(K)$ is compact for every compact subset K of $\mathbb{R}$, is f continuous on $\mathbb{R}$?

2. (15 points) Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k$ converges uniformly on [0,1].

3. (15 points) Define

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \end{cases}$$

   Is $f(x,y)$ differentiable at (0,0)? Show your reason.

4. (15 points) Let $f : (-1,2) \to \mathbb{R}$ be a real analytic function. If $f(\frac{1}{k}) = 0$ for all $k \in \mathbb{N}$, show that f is identically zero.

5. (15 points) Let f be a continuous function on $K = [0,1] \times [0.1] \subset \mathbb{R}^2$, show that there exists an interior point (x_0, y_0) of K such that

$$\iint_K f(x,y)\, dx\, dy = f(x_0, y_0).$$

6. (15 points) Let f be a continuous, bounded function on $\mathbb{R}$, and define $g(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy$.
   (a) Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.
   (b) Show that $(1 - \frac{d^2}{dx^2})g = f$.

7. (15 points) Show that the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

   is uniformly continuous on $\mathbb{R}$.

8. (15 point) Evaluate the line integral

$$I = \oint_C \frac{x dy - y dx}{(x+2y)^2 + (3x-y)^2},$$

   where C is the closed curve defined by $(x+2y)^2 + (3x-y)^2 = 1$ in counterclockwise direction.