Algrbra and Linear Algebra

15% (1) Let $\mathbb{Q}[x, y]$ be the polynomial ring in two variables with rational coefficients. Let $I \subset \mathbb{Q}[x, y]$ be the ideal generated $x-2$, and $y-3$. Prove that I is a maximal ideal in $\mathbb{Q}[x, y]$.

15% (2) Let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. Given non-zero integers m, n, and d, let $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the homomorphism of abelian groups defined by

$$(a, b) \mapsto (ma + db, nb).$$

Compute the index $(\mathbb{Z}^2 : f(\mathbb{Z}^2))$. Explain your answer.

15% (3) Let α and β be algebraic numbers. Suppose $\mathbb{Q}(\alpha)$ has degree m over $\mathbb{Q}$, and $\mathbb{Q}(\beta)$ has degree n over $\mathbb{Q}$, with m and n relatively prime. What is the degree of the field $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$? Prove your answer.

15% (4) The number of elements in a finite set S is denoted by $\#S$. Let A, B be subgroups of a finite group G. Given $x \in G$, the subset $AxB = \{axb | \ a \in A, b \in B\} \subset G$ is called a double coset of A and B in G. Prove the following:

(a) G is a disjoint union of double cosets of A and B in G.

(b)

$$\#(AxB) = \frac{\#A \ \#B}{\#(A \cap xBx^{-1})}.$$

15% (5) Let T be a linear operator on a finite dimensional vector space V. Suppose T is idempotent; that is, $T^2 = T$. Prove that

(a) Eigenvalues of T are either 0 or 1.

(b) $V = \text{Kernel}(T) \oplus \text{Range}(T)$.

(c) T is diagonalizable.

15% (6) Let $P_3(\mathbb{C})$ be the space of complex polynomials of degree less than or equal to 3. Define the linear operator T on $P_3(\mathbb{C})$ by

$$T(f) = -\frac{x^2}{2}f''' + f'' + f' - 2f.$$

Find the Jordan canonical form and the minimal polynomial of T.

15% (7) Let T be a self-adjoint operator (i.e. $T = T^*$) on $\mathbb{C}^n$.

(a) Show that all eigenvalues of T are real.

(b) Suppose λ, μ are distinct eigenvalues of T and E_λ, E_μ are the corresponding eigenspaces. Show that $E_\lambda \subset E_\mu^\perp$.

15% (8) Let T and U be linear operators on a finite-dimensional vector space V. Suppose that T commutes with U, i.e. $TU = UT$. Prove that if T has an eigenvector and U has an eigenvector then there is a vector $v \in V$ which is an eigenvector for both T and U.