國 立 清 華 大 學 命 題 紙

94學年度＿數 學＿系（所）＿純數＿組碩士班入學考試

科目＿高等微積分＿科目代碼＿0101＿共＿1＿頁第＿1＿頁 *請在試卷【答案卷】內作答

Show your work, otherwise no credit will be granted.

1. (15 points) Let f be a continuous real-valued function defined on $[a,b]$, and let $M = \max_{x\in[a,b]} |f(x)|$. Show that $\lim_{n\to\infty}(\int_a^b |f(x)|^n\, dx)^{1/n} = M$.

2. (15 points) Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k$ converges uniformly on [0,1].

3. (15 points) Let $f : (-1,2) \to \mathbb{R}$ be a real analytic function. If $f(\frac{1}{k}) = 0$ for all $k \in \mathbb{N}$, show that f is identically zero.

4. (15 points) Let f be a nonnegative real-valued function defined on [0,1]. Suppose that there is an universal constant $M > 0$ such that $f(x_1)+\cdots+f(x_k) \le M$ for every finite subset $\{x_1, \cdots, x_k\}$ of [0,1]. Show that the set $S = \{x \in [0,1] \mid f(x) \neq 0\}$ is countable.

5. (15 points) Let f be a continuous function on $K = [0,1] \times [0.1] \subset \mathbb{R}^2$, show that there exists an <u>interior point</u> (x_0, y_0) of K such that

$$\iint_K f(x,y)\, dx\, dy = f(x_0, y_0).$$

6. (15 points) Show that the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

is uniformly continuous on $\mathbb{R}$.

7. (15 point) Prove that

$$\lim_{n\to\infty} \int_0^1 (1+\frac{x}{n})^n e^{-x} dx = 1.$$

8. (15 points) Consider the normed space $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ with sup norm $\| f \|_\infty = \max_{x\in[0,1]} |f(x)|$. Let $B = \{f \in C([0,1]) : \int_0^1 f(x)^2 dx < 1\}$.

   (a) Is B open? Prove your answer.

   (b) Is B connected? Prove your answer.