# Linear Algebra Exam

1. (15%) Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & 1 & -1 & 3 & -4 \end{pmatrix}.$$

Find a $5 \times 5$ real matrix $M$ with $rank(M) = 2$ such that $AM = 0$, where $0$ is the $4 \times 5$ zero matrix.

2. (15%) Let $L$ be the line in $\mathbb{R}^3$ defined by the linear equations

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0. \end{cases}$$

Let $H$ be the plane in $\mathbb{R}^3$ that passes through $(0,0,0)$ and has $L$ as its normal line. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that fixes $L$ and is a rotation by 180° in the plane $H$. Find the matrix representation $[T]_\beta$ with respect to the standard basis $\beta = \{e_1, e_2, e_3\}$.

3. (15%) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a diagonalizable linear transformation. Prove that $\mathbb{R}^4$ is $T$-cyclic (means there exists a $v \in \mathbb{R}^4$ such that $\{v,\ T(v),\ T^2(v),\ T^3(v)\}$ is a basis of $\mathbb{R}^4$) if and only if each eigenspace of $T$ is one-dimensional.

4. (18%) Let $V$ be the real vector space of all $2\pi$-periodic smooth functions $f : \mathbb{R} \to \mathbb{R}$. Define an inner product $\langle , \rangle$ on $V$ by

$$\langle f,\ g \rangle = \int_0^{2\pi} f(x)\, g(x)\, dx, \qquad f,\ g \in V.$$

Let $T : V \to V$ be a linear operator defined by

$$T[f(x)] = \frac{d^2 f}{dx^2}(x) + f(x), \qquad f \in V.$$

Assume that $T$ has an adjoint operator $T^* : V \to V$.

(a) Show that $T : (V,\ \langle , \rangle) \to (V,\ \langle , \rangle)$ is self-adjoint.

(b) Show that the differential equation

$$\frac{d^2 f}{dx^2}(x) + f(x) = \cos x, \qquad x \in [0, 2\pi]$$

has no solutions $f(x)$ in $V$.

5. (12%) Let $A$ be an $n \times n$ real symmetric matrix. It is known that there exists an orthonormal basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of $A$. Let

$$R(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle}, \quad x \in \mathbb{R}^n, \quad x \neq 0.$$

Show that $\max_{x\in\mathbb{R}^n, x\neq 0} R(x)$ is the largest eigenvalue of $A$ and $\min_{x\in\mathbb{R}^n, x\neq 0} R(x)$ is the smallest eigenvalue of $A$. Here $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$.

6. (15%) Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & 1 \end{pmatrix}$$

where $a,\ b,\ c \in \mathbb{R}$. Find conditions on $a$, $b$ and $c$ such that $A$ is diagonalizable. Give your reasons.

7. (10%) True or false.

(a) Let $(V, \langle , \rangle)$ be a finite dimensional real inner product space and let $T : V \to V$ be a self-adjoint linear operator. Then with respect to any ordered basis $\beta$ of $V$, the matrix $[T]_\beta$ is symmetric.

(b) Let $T : V \to W$ be a linear transformation such that $T$ carries each linearly independent subset of $V$ onto linearly independent subset of $W$, then (null space of $T$) $= \{0\}$.

(c) If the coefficient matrix of a system of $m$ linear equations $AX = b$ in $n$ unknowns has rank $m$, then the system has a solution.

(d) Let $T : V \to V$ be a linear operator on a real vector space $V$, $\dim V = n$. Assume the characteristic polynomial $f(x)$ of $T$ has $n$ real roots, then $T$ is diagonalizable if and only if, for each eigenvalue $\lambda$ of $T$, the multiplicity of $\lambda$ equals $n - rank\,(T - \lambda I)$.

(e) If $B$ is a matrix obtained from a square matrix $A$ by adding $k$ times row $i$ to row $j$, then $\det(B) = k \det(A)$.