國 立 清 華 大 學 命 題 紙

九十三學年度 數學 系（所） 應數 組碩士班入學考試

科目 高等微積分 科號 0201 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

Show your work, otherwise no credit will be granted.

(1). (15 points) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of real-valued functions of bounded variation on $[a,b]$ with variation $V(f_k;a,b) \le M < \infty$ for all k and some $M > 0$. If f_k converges pointwise to a function f on $[a,b]$, show that f is of bounded variation on $[a,b]$ and that $V(f;a,b) \le M$.

(2). (15 points) Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Suppose that $f^{-1}(C)$ is connected for every connected subset C of $\mathbb{R}$. Is f continuous on $\mathbb{R}$? Prove it or give a counterexample.

(3). (15 points) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Let $s_n = a_1 + \cdots + a_n$ be the partial sum of the series.

(i) Does $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ converge? Prove or disprove it.

(ii) Does $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converge? Prove or disprove it.

(4). (15 points) Let $f(x,y)$ be a function defined on $\mathbb{R}^2$. Suppose that $f(x,y)$ is real analytic in x if y is fixed and that $f(x,y)$ is real analytic in y if x is fixed. Is this function f differentiable at (0,0)? Prove it or give a counterexample.

(5). (15 points) Let f be a real-valued, differentiable function on $\mathbb{R}$ such that $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Assume that $f(0) = 0$, show that $f(x) > 0$ for all $x > 0$.

(6). (15 points) Find the extrema of the function $f(x,y,z) = x - y + z$ on the domain $\{(x,y,z) : 1 - x^2 - y^2 \ge z \ge x^2 + y^2, x \ge 0\}$.

(7). (15 points) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued, differentiable functions defined on [0,3] such that $f_n(1) = 1$ for all n and $|f'(x)| \le 5$ for all x and all n. Show that there exists a subsequence of $\{f_n\}$ which converges uniformly on [0,3].

(8). (15 points) Evaluate the surface integral $\iint_S (xy + xz + yz)d\sigma$, where $d\sigma$ is the surface element and S is a portion of the cone $\{(x,y,z) : x^2 + y^2 = z^2, z \ge 0\}$ inside the cylinder $\{(x,y,z) : x^2 + y^2 - 2x = 0\}$.