國 立 清 華 大 學 命 題 紙

九十三學年度 數學 系（所） 純數 組碩士班入學考試

科目 代數及線性代數 科號 0102 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

Algebra & Linear Algebra Exam

1. (16%)

   (a) Describe all subgroups of order 4 of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

   (b) How many such subgroups are cyclic? How many such subgroups are non-cyclic? Give your reasons.

2. (18%)

   (a) Find a maximal ideal of the ring $\mathbb{Z} \times \mathbb{Z}$.

   (b) Find a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not a maximal ideal.

3. (16%) Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & 1 & -1 & 3 & -4 \end{pmatrix}.$$

   Find a 5×5 real matrix M with $rank(M) = 2$ such that $AM = 0$, where 0 is the 4×5 zero matrix.

4. (18%) Let L be the line in $\mathbb{R}^3$ defined by the linear equations

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0. \end{cases}$$

   Let H be the plane in $\mathbb{R}^3$ that passes through $(0,0,0)$ and has L as its normal line.
   Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that fixes L and is a rotation by 180° in the plane H. Find the matrix representation $[T]_\beta$ with respect to the standard basis $\beta = \{e_1, e_2, e_3\}$.

5. (16%) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a diagonalizable linear transformation. Prove that $\mathbb{R}^4$ is T-cyclic (means that there exists a $v \in \mathbb{R}^4$ such that $\{v, T(v), T^2(v), T^3(v)\}$ is a basis of $\mathbb{R}^4$) if and only if each eigenspace of T is one-dimensional.

6. (16%) Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$ of degree 25. Let $\mathbb{Q}(\alpha^3)$ be the subfield of $\mathbb{C}$ by adjoining α^3 to $\mathbb{Q}$. Prove that $\alpha \in \mathbb{Q}(\alpha^3)$.

7. (20%) Let G be the set of all $n \times n$ upper triangular real matrices with non-zero diagonal elements. It is a group under matrix multiplication (you do not have to prove this). Let H be the subgroup of G consisting of those matrices whose diagonal elements are all equal to 1.

   (a) Prove that H is a normal subgroup of G.

   (b) Prove that the factor group G/H is isomorphic to the direct product $\mathbb{R}^* \times \cdots \times \mathbb{R}^*$ (n times) where $\mathbb{R}^* = \mathbb{R} - \{0\}$ is the group under multiplication in $\mathbb{R}$.