國 立 清 華 大 學 命 題 紙

九十三學年度 數學 系（所） 純數 組碩士班入學考試

科目 高等微積分 科號 0101 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

Show your work, otherwise no credit will be granted.

(1). (15 points) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of real-valued functions of bounded variation on $[a,b]$ with variation $V(f_k;a,b) \leq M < \infty$ for all k and some $M > 0$. If f_k converges pointwise to a function f on $[a,b]$, show that f is of bounded variation on $[a,b]$ and that $V(f;a,b) \leq M$.

(2). (15 points) Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Suppose that $f^{-1}(C)$ is connected for every connected subset C of $\mathbb{R}$. Is f continuous on $\mathbb{R}$? Prove it or give a counterexample.

(3). (15 points) Prove that

$$\left|\int_0^1 x\sin\frac{1}{x}\,dx\right| \leq \left(\int_0^1 x^2\sin^2\frac{1}{x}\,dx\right)^{1/2}.$$

(4). (15 points) Let f be a real-valued function defined on [0,1]. Suppose that $\{x \in [0,1] \mid f(x) < a\}$ is an open subset of [0,1] for every real number a. Show that f assumes its maximum on [0,1], i.e., that there exists $x_0 \in [0,1]$ such that $f(x_0) \geq f(x)$ for all $x \in [0,1]$.

(5). (15 points) Let f be a real-valued, differentiable function on $\mathbb{R}$ such that $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Assume that $f(0) = 0$, show that $f(x) > 0$ for all $x > 0$.

(6). (15 points) Find the extrema of the function $f(x,y,z) = x - y + z$ on the domain $\{(x,y,z) : 1 - x^2 - y^2 \geq z \geq x^2 + y^2, x \geq 0\}$.

(7). (15 points) Let f be a continuous function on $\mathbb{R}$ such that $f(x) = f(x+1)$ for all x. Show that (i) f takes both maxima and minima, and (ii) there is a x_0 such that $f(x_0 + \frac{1}{2}) = f(x_0)$.

(8). (15 points) Let $C = \{(2\cos t, \sin t) : 0 \leq t \leq \pi\}$ and $\mathbf{F} = x^2(y^3+1)\mathbf{i} + x(x^2y^2+1)\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.