國 立 清 華 大 學 命 題 紙

九十二學年度 數 學 系（所） 純粹數學 組碩士班研究生招生考試

科目 代數及線性代數 科號 0102 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (20 points) Find the value of c so that the system of linear equations $\begin{cases} x+y+z=1 \\ x-y+z=6 \\ x+5y+z=c \end{cases}$ has solutions in $\mathbb{R}^3$, and in that case, find all the solutions.

2.(16 points) Let G be a finite group with order $|G| = p^4q$, where p, q are distinct primes. Suppose that its center $Z(G)$ satisfies $|Z(G)| = q$.

(a) Prove that any p-Sylow subgroup H of G is isomorphic to $G/Z(G)$.

(b) Prove that the group G is isomorphic to the direct product $H \times Z(G)$.

3.(22 points) (a) Let A be an $m \times n$ real matrix, B an $n \times p$ real matrix. Prove that $rank(AB) \geq rankA + rankB - n$.

(b) Use (a) to show that if $A_1, ..., A_k$ are $n \times n$ real matrices satisfying $A_1 \cdots A_k = 0$, then $rankA_1 + ... + rankA_k \leq (k-1)n$.

4.(20 points) Let H be a group and let $A = Aut(H)$, the group of all automorphisms of H under composition. Define a product on the set $G = \{(f,x) : f \in A, x \in H\}$by $(f,x) \cdot (g,y) = (f \circ g, g^{-1}(x) \cdot y)$.

(a) Prove that G is a group under this product. and that $\{(id,x) : x \in H\}$ is a normal subgroup of G.

(b) For $H = \mathbb{Z}_4$, find $Aut(\mathbb{Z}_4)$. Describe the group G above for $H = \mathbb{Z}_4$by finding a minimal set of generators and giving the relations among them.

5.(12 points) If a is a complex number satisfying $p(a) = 0$, where $p(x) = x^3 + x + \sqrt{2}$, prove that a is algebraic over $\mathbb{Q}$ (field of rational numbers) of degree 6.

6.(30 points) For each of the following statements, sketch a proof if it is true, explain why or give a counterexample if it is false.

(a) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation satisfying $T^4 = -I$, then n has to be even.

(b) Let A, B be real symmetric $n \times n$ matrices, then there exists a nonsingular matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

(c) If A is a singular $n \times n$ real matrix, then there exists a nonzero $n \times n$ matrix B satisfying $BA = 0$.

(d) 13 is irreducible in the ring $\mathbb{Z}[\sqrt{3}] = \{m + n\sqrt{3} : m, n \in \mathbb{Z}\}$ (regarded as a subring of the field $\mathbb{R}$ of real numbers).

(e) The quotient field of $D = \{2^k b : k, b$ are integers; b is 0 or odd$\}$ (regarded as a subring of $\mathbb{Q}$) is $\mathbb{Q}$.