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九十二學年度 數 學 系（所） 純粹數學 組碩士班研究生招生考試

科目 高等微積分 科號 0101 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

*** Show your work, otherwise no credit will be granted.**
****Each problem is worth 15 points.**

1. Let f and g be two continuous real-valued functions on [0,1] such that $\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x)$. Show that there is a $t \in [0,1]$ such that $f(t) = g(t)$.

2. For any subsets A, B of $\mathbb{R}^2$, define $A + B = \{c \mid c = a + b, a \in A, b \in B\}$.
   (1) If A is closed and B is compact, show that $A + B$ is closed.
   (2) As in (1), but B is assumed to be closed only, is $A + B$ closed? Prove it or give a counterexample.

3. Let $f_n(x) = \cos(nx)$ on $[0, \pi]$, does the sequence $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence? Prove or disprove it.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative continuous functions on [0,1] such that $f_n(x) \ge f_{n+1}(x)$, $n = 1, 2, \cdots$, for all $x \in [0,1]$. Let $f(x) = \lim_{n\to\infty} f_n(x)$ and $M = \sup_{0\le x\le 1} f(x)$, show that there is a $t \in [0,1]$ such that $f(t) = M$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
   (1) If f is continuous in the ϵ-δ sense, show that $f^{-1}(C)$ is closed for any closed subset C of $\mathbb{R}$.
   (2) If $f^{-1}(K)$ is compact for any compact subset K of $\mathbb{R}$, is f continuous on $\mathbb{R}$? Prove it or give a counterexample.

6. Define

$$f(x,y) = \begin{cases} (x-y)^2 \sin\frac{1}{x-y}, & \text{if } x \ne y, \\ 0, & \text{if } x = y. \end{cases}$$

Is f differentiable at (0,0)? Prove or disprove it.

7. Evaluate the surface integral over the sphere $S = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$:

$$\iint_S (x^4 + y^4)\, d\sigma,$$

where $d\sigma$ is the surface element.

8. Let f be a C^2 real-valued function on $(0, \infty)$, and let $M_0 = \sup|f(x)|$, $M_1 = \sup|f'(x)|$, $M_2 = \sup|f''(x)|$ on $(0, \infty)$. Show that $M_1^2 \le 4M_0M_2$.