1. (15 points) Evaluate the integral

$$\int_0^1 \int_y^{\sqrt{y}} e^{3x^2-2x^3} dx dy.$$

2. (15 points) Evaluate the integral

$$\int_0^\infty e^{-7x^2} dx.$$

3. (15 points) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function. Suppose that

$$x \cdot \nabla f(x) = k f(x).$$

Prove that

$$f(\lambda x) = \lambda^k f(x) \text{ for each } x \in \mathbf{R}^n \text{ and } \lambda > 0.$$

4. (15 points) How many real numbers x satisfy

$$x + 1 = 4 \arctan(x)?$$

5. (15 points)

   (a) Let f be a continuous function from $[0,1]$ onto $[0,1]$, show that there exists a point $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

   (b) If g is a continuous function from $(0,1)$ onto $(0,1)$, can there always exists a fixed point in $(0,1)$ for g? Justify your answer.

6. (15 points) Suppose $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in [a, b]$ and for each n, f_n satisfies

$$|f_n(x) - f_n(y)| \le M|x - y|, \text{ for all } x, y \in [a, b], \tag{1}$$

where M is independent of n. Show that f also satisfies the condition (1).

7. (15 points) Define

$$f(x,y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Show that f has directional derivatives in all directions at $(0,0)$.

(c) Show that f is not differentiable at $(0,0)$.

8. (15 points) Check whether the following sets E in $\mathbf{R}^2$ are compact or not. If E is not compact, find the smallest compact set K such that $E \subset K$. Justify your answers!

(a) $E = \{(x,y) : y = \sin(\frac{1}{x}), x \in (0,1]\}$.

(b) $E = \{(x,y) : x \in [0,1]$ and $y = 1$ if x is rational, $y = 0$ if x is irrational$\}$.