1. (15 points) Find the maximum and minimum of $xy+z^2$ on the set $\{(x,y,z): x^2+y^2+z^2 \le 1\}$.

2. (15 points) Let f be a one to one continuous function on $[0,1]$. Show that f is either strictly increasing or strictly decreasing.

3. (15 points) If $C \subset \mathbf{R}^n$ is connected, show that its closure $cl(C)$ is also connected.

4. (15 points) Let $f: [0,1] \to \mathbf{R}$ be a continuous function, consider the sequence of functions

$$f_0(x) = f(x),\ f_{n+1}(x) = \int_0^x f_n(t)dt,\ n = 0,1,2,3,\cdots,\ x \in [0,1].$$

Show that $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $[0,1]$.

5. (15 points) Let $f: [a,b] \to \mathbf{R}$ be continuous, and

$$F(x) \equiv \int_a^b f(y)|x-y|dy.$$

Find $F''(x)$.

6. (15 points) If $f: \mathbf{R}^2 \to \mathbf{R}$ has continuous second derivatives and if $g(r,\theta) = f(r\cos\theta, r\sin\theta)$, $r>0$, $\theta \in \mathbf{R}$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2},$$

where $x = r\cos\theta$, $y = r\sin\theta$.

7. (15 points) A real value function $f(x)$ on (a,b) is a convex function if

$$f(\lambda c + (1-\lambda)d) \le \lambda f(c) + (1-\lambda) f(d)$$

for all $a<c<d<b$ and $0 \le \lambda \le 1$. Prove that f is a differentiable convex function on (a,b) iff $f'(x)$ is increasing on (a,b).

8. (15 points) Let $S = \{(x,y,z): x^2+y^2+z^2 = 1,\ z \ge 0\}$ and $\overrightarrow{F} = y^2z^3\overrightarrow{i} + x^4z\overrightarrow{j} + (x+y)\overrightarrow{k}$. Evaluate the surface integral

$$\iint_S \overrightarrow{F} \bullet \overrightarrow{n}\, dS$$

where $\overrightarrow{n}$ is the unit normal vector of S pointing outward.