

- (10 points) Show that $|\cos^2 x - \cos^2 y| \leq |x - y|$ for any $x, y \in \mathbb{R}$.
- (10 points) Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable function, and let S be a surface defined by $S = \{(x, y, z) : f(x, y, z) = c\}$ for some constant c . Prove that $\nabla f(x, y, z)$ is perpendicular to S if $(x, y, z) \in S$.
- (16 points) Suppose $f: (-5, 5) \rightarrow \mathbb{R}$ is a continuous function. Which of the following statements must be true and which could be false? (Give reasons for your answers)
 - The set $\{f(x) : 0 < x < 1\}$ is open.
 - The set $\{f(x) : 0 < x < 1\}$ is bounded.
 - f is uniformly continuous on the interval $(-1, 1)$.
- (12 points) Suppose $f: [0, 1] \rightarrow \mathbb{R}$ is a continuous function such that $\int_0^1 f(x) dx = \frac{1}{2}$. Show that there exists a point $p \in (0, 1)$ such that $f(p) = p$.
- (12 points) Show that the continuous function $f(x, y) = (xy)^{1/3}$ has partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$, but f is not differentiable at $(0, 0)$.
- (12 points) Find

$$\lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n}$$

by considering Riemann sums for $\int_0^1 \log x dx$ based on the partition $\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$.

- (12 points) Let ϵ be a positive number. Show that there is no function f satisfying the following conditions: $f'(x)$ exists for $x \geq 0$, $f'(0) = 0$ and $f'(x) \geq \epsilon$ for $x > 0$.
- (12 points) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(x-n)^2}$$

is well-defined and continuous on $\mathbb{R} \setminus \{0, 1, 2, \dots\}$.

9. (12 points) Evaluate the line integral

$$\oint_{\partial D} \frac{dx + dy}{|x| + |y|},$$

where D is the rectangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

10. (12 points) Consider the family of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$F_t(x, y) = (x + ty^2, x + y), \quad t \in \mathbb{R}.$$

Let $C = [0, 1] \times [0, 1]$, and set $C_t = F_t(C)$.

- a) Show that the set $\bigcup_{t \in \mathbb{R}} C_t$ is connected.
b) Is the set $\bigcup_{t \in \mathbb{R}} C_t$ compact? Explain.