

國 立 清 華 大 學 命 題 紙

八十八學年度 數學系 系(所) 應用數學組碩士班研究生招生考試

科目 高等微積分 科號 D201 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (10 points) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x)$ is rational for all x . Prove that f is a constant function.
2. (10 points) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable such that $f'(x) \neq 0$ for all x . Prove that f is one-to-one.
3. (10 points) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Suppose $f(a) = 0$, $f(b) = 1$, and $\int_a^b f(x) dx = 0$. If f is differentiable in (a, b) , prove that there is a point $c \in (a, b)$ such that $f'(c) = 0$.

4. (10 points) Consider the line integral

$$I_R = \oint_{x^2+y^2=R^2} \frac{xdy - ydx}{(x^2 + xy + y^2)^2}$$

Find the limit $\lim_{R \rightarrow \infty} I_R$.

5. (16 points) Find the extrema of $x^2(1-x) + y^2(1-y) + z^2(1-z)$ subject to the constraints: $x + y + z = 1$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.

6. (16 points) Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f_n(x) = nx(1-x^2)^n, \quad n = 1, 2, 3, \dots$$

(a) Compute $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ and $\int_0^1 [\lim_{n \rightarrow \infty} f_n(x)] dx$.

(b) Does $\{f_n\}$ converge uniformly on $[0, 1]$?

7. (16 points) For what values of λ , does the integral

$$\iiint_{\mathbb{R}^3} \frac{dV}{(x^2 + y^2 + z^2)^\lambda (1 + x^2 + y^2 + z^2)}$$

converge? Show your works.

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8. (16 points) Let $\bar{B}(0, r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$. Suppose $f : \bar{B}(0, r) \rightarrow \mathbb{R}^n$ is a map with $\|f(0)\| \leq \frac{1}{3}r$ and $\|f(x) - f(y)\| \leq \frac{2}{3}\|x - y\|$ for all $x, y \in \bar{B}(0, r)$. Prove that there is a unique point $p \in \bar{B}(0, r)$ such that $f(p) = p$.
9. (16 points) Show that the equation

$$x \sin y = y + \sin x$$

has a solution of the form $y = f(x)$ for (x, y) near $(0, 0)$. Find the first three terms in the Taylor expansion of $f(x)$ about $x = 0$.