1. (10 points) Let G be a group of order 257. Show that G is cyclic.

2. (10 points) Let ω be a primitive 5th root of unity and let

   $f(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ be a polynomial with rational coefficients.

   If $f(\omega) = 0$, prove that $a_3 = a_2 = a_1 = a_0 = 1$.

3. (15 points) Construct a field having 125 elements. Can we construct a field with 123 elements?

4. (20 points) Let $\mathbf{Q}$ be the set of all rational numbers. Let G be the group of all matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

   where $a, b, c \in \mathbf{Q}$, under matrix multiplication.

   (a) Find the center C of G and show that C is isomorphic to the additive group $\mathbf{Q}$.

   (b) Show that G/C is isomorphic to the additive group $\mathbf{Q} \times \mathbf{Q}$.

5. (15 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map such that $R(T) = N(T)$, where $R(T) = \{Tx \mid x \in \mathbf{R}^2\}$ and $N(T) = \{x \in \mathbf{R}^2 \mid Tx = 0\}$. Find $\mathrm{tr}(T)$ and $\det(T)$.

6. (20 points) Let D_n be the determinant of the 1, 1, 1 tridiagonal $n \times n$ matrix

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 1 \end{pmatrix}.$$

   Evaluate the value of D_n for each positive integer n.

7. (15 points) Let $A = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$.

   (a) Find the the minimal polynomial of A.

   (b) Find A^m, where m is a positive integer.

8. (15 points) Let $\Gamma = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \right\}$ and $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Find the area of the region enclosed by $A(\Gamma) = \left\{ A \begin{pmatrix} x \\ y \end{pmatrix} \mid \begin{pmatrix} x \\ y \end{pmatrix} \in \Gamma \right\}$.