1. (12 points) Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and $f(x)$ is rational for all x. Prove that f is a constant function.

2. (12 points) Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable and suppose there is a constant $M > 0$ such that $|f'(x)| \le M$ for all x. Prove that f is uniformly continuous on $\mathbf{R}$.

3. (12 points) Let $f : [a, b] \to \mathbf{R}$ be continuous. Suppose $f(a) = 0$, $f(b) = 1$, and $\int_a^b f(x)dx = 0$. If f is differentiable in (a, b), prove that there is a point $c \in (a, b)$ such that $f'(c) = 0$.

4. (12 points) Consider the line integral

$$I_R = \oint_{x^2+y^2=R^2} \frac{x^2 dy - y^2 dx}{x^2 + xy + y^2}.$$

Find the limit $\lim_{R \to 0} I_R$.

5. (12 points) If $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous, prove that f is not one-to-one.

6. (15 points) Let $f_n : [0, 1] \to \mathbf{R}$ be defined by

$$f_n(x) = nx(1 - x^2)^n, \quad n = 1, 2, 3, \cdots.$$

   (a) Compute $\lim_{n \to \infty} \int_0^1 f_n(x)dx$ and $\int_0^1 [\lim_{n \to \infty} f_n(x)]dx$.

   (b) Does $\{f_n\}$ converge uniformly on $[0, 1]$?

7. (15 points) For what values of λ, does the integral

$$\iiint_{\mathbf{R}^3} \frac{dV}{(x^2 + y^2 + z^2)^\lambda (1 + x^2 + y^2 + z^2)}$$

converge? Show your works.

8. (15 points) Let $\overline{B}(0, r) = \{x \in \mathbf{R}^n : \|x\| \le r\}$. Suppose $f : \overline{B}(0, r) \to \mathbf{R}^n$ is a map with $\|f(0)\| \le \frac{1}{3}r$ and $\|f(x) - f(y)\| \le \frac{2}{3}\|x - y\|$ for all $x, y \in \overline{B}(0, r)$. Prove that there is a unique point $p \in \overline{B}(0, r)$ such that $f(p) = p$.

9. (15 points) Show that the equation

$$y \sin x = x + \sin y$$

has a solution of the form $y = f(x)$ for (x, y) near $(0, 0)$. Find the first three terms in the Taylor expansion of $f(x)$ about $x = 0$.