國 立 清 華 大 學 命 題 紙

八十七學年度 數學系 系(所)應用數學組碩士班研究生入學考試

科目 微分方程 科號 0203 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15 points)

   (a) The equation $\frac{dp}{dt} = ap^{\alpha}$, $\alpha > 1$, $a > 0$, $p(0) = p_0 > 0$, is proposed as a model of population growth of a certain species. Show that $p(t) \to \infty$ in finite time. Hence it is not an appropriate model.

   (b) If we modify the model to $\frac{dp}{dt} = ap^{\alpha} - bp$, $b > 0$, discuss the behavior of the solution $p(t)$ as $t \to \infty$.

2. (10 points) Solve the following integral equation

$$y(x) + \int_0^x (x-t)y(t)dt = x^2$$

3. (15 points) Consider the following system of differential equations

$$\frac{dx}{dt} = -y + x(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right)$$
$$\frac{dy}{dt} = x + y(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right)$$
$$(x(0), y(0)) \neq (0,0)$$

Use polar coordinate (r,θ), $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$ to derive $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ and discuss the behavior of the solution $(x(t), y(t))$ as $t \to \infty$.

4. (10 points) The Bernoulli equation is of the following form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 0, 1$$

   (a) Let $v = y^{1-n}$. Reduce Bernoulli equation to linear equation.

   (b) Solve $\begin{cases} \frac{dy}{dx} = \epsilon y - \sigma y^3, \ \epsilon > 0, \ \sigma > 0 \\ y(0) = 1 \end{cases}$

5. (10 points)

   (a) Show that the solution of initial-value problem

$$y'' + y = g(x), \quad y(x_0) = 0, \quad y'(x_0) = 0$$

   is

$$y(x) = \int_{x_0}^x \sin(x-t)g(t)dt$$

   (b) Find the solution of initial value problem

$$y'' + y = g(x), \quad y(0) = y_0, \quad y(0) = y_0'$$

6. (10 points)

(a) Let $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$ be the Laplace transform of $f(t)$. Suppose $\frac{f(t)}{t}$ has a limit as $t \to 0$. Prove that

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)du$$

(b) Find the Laplace transform of $\frac{\sin t}{t}$

7. (10 points) Solve the second-order linear difference equation

$$\begin{cases} x_{n+2} = x_{n+1} + x_n \\ x_0 = 0, \ x_1 = 1 \end{cases}$$

(Hint: Consider the general solution $x_n = \lambda^n$ for some λ)