國立清華大學命題紙

八十七學年度 數學系 系（所）純粹數學組碩士班研究生入學考試

科目 高等微積分 科號 0101 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1(15 pts). (a) Evaluate the double integral

$$\int_0^1 \int_y^1 e^{x^2}\, dx dy.$$

(b) Define

$$g(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 3x - 1 & \text{if } 0 < x < 1 \\ 2 & \text{if } x \ge 1 \end{cases} \quad \text{and} \quad \alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}.$$

Evaluate the Riemann-Stieltjes integral

$$\int_{-2}^{2} g(x)\, d\alpha(x).$$

2(10pts). Evaluate the surface integral

$$\int\int_{\sum} (x^4 + y^4 + z^4)\, d\sigma$$

where $\sum$ is the unit sphere in $\mathbf{R}^3$ and $d\sigma$ is the surface element on $\sum$.

3(20pts). Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} \sqrt{x^2 + y^2}\, \sin\left(\frac{y^2}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) Show that f is continuous at the point $(0,0)$.
(b) Show that f has directional derivatives in every direction at $(0,0)$.
(c) Is f differentiable at $(0,0)$? Explain.

4(15 pts). (a) Show that the series $\sum_{k=1}^{\infty} \frac{\cos^2 kx}{k\sqrt{k}}$ converges uniformly on $\mathbf{R}$.
(b) Does there exist a polynomial $p(x)$ such that

$$\left| p(x) - \sum_{k=1}^{\infty} \frac{\cos^2 kx}{k\sqrt{k}} \right| < 10^{-4} \quad \text{for all } x \in (-1,1)\ ?$$

Show your reason.

5(10 pts). Let $f : \mathbf{R} \to \mathbf{R}$ be a bounded function. Does there exist a sequence of positive integers $n_1 < n_2 < n_3 < \cdots \to \infty$ such that $\lim_{k\to\infty} f(n_k)$ exists? Show your reason.

6(15 pts). Let $S = \{(x,y) : x^2 + y^2 = 1\}$ be the unit circle in $\mathbf{R}^2$, and let $f : S \to \mathbf{R}$ be a continuous function. Prove that there are two antipodal points (x_0, y_0) and $(-x_0, -y_0)$ in S such that $f(x_0, y_0) = f(-x_0, -y_0)$.

7(20 pts). (a) Let $h : \mathbf{R} \to \mathbf{R}$ be a differentiable function, and suppose that there is a constant $c > 0$ such that $h'(t) \geq c$ for all $t \in \mathbf{R}$. Prove that there is exactly one point t at which $h(t) = 0$.
(b) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a C^1 function, and suppose that there is a constant $c > 0$ such that

$$\frac{\partial f}{\partial y}(x,y) \geq c \quad \text{for all } (x,y) \in \mathbf{R}^2.$$

Prove that there is a C^1 function $g : \mathbf{R} \to \mathbf{R}$ with $f(x, g(x)) = 0$ for all $x \in \mathbf{R}$.

8(15 pts). Let $f(x,y)$ be a continuous real-valued function defined on the closed unit disc $\overline{\Delta}$, where $\Delta = \{(x,y) : x^2 + y^2 < 1\}$. Suppose that f satisfies the submean value property on Δ, i.e., for any $p = (x_0, y_0) \in \Delta$ and any $0 < r < 1 - \sqrt{x_0^2 + y_0^2}$, we have

$$f(p) \leq \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta.$$

Prove that if f is not a constant function, then f must achieves its maximum on the boundary of Δ.