八十六學年度 數學系 系(所) 應數 組碩士班研究生入學考試

科目 微 分 方 程 科號 0203 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15 points)
Find the solution of the following initial value problem

$$y''(x) = 1 - (y'(x))^2$$
$$y(0) = 0,\ y'(0) = 0$$

2. (20 points)
Consider the following initial value problem

$$x'(t) = x + e^{-x},\ x(0) = 0$$

(a) Show that the solution $x(t)$ is positive for $t > 0$; $x(t)$ is strictly increasing and concave upward for $t > 0$ and $x(t) \le e^t - 1$ for $t \ge 0$.

(b) Show that $\lim_{t\to\infty} x(t) = +\infty$.

3. (10 points)
Let a, b, c be positive constants and d is a real constant. If $y(t)$ is a solution of $ay'' + by' + cy = d$, show that $\lim_{t\to\infty} y(t) = d/c$.

4. (20 points)

(a) Use the method of Laplace transforms to solve the initial value problem

$$(*) \qquad y''(t) + y(t) = g(t)$$
$$y(0) = 0,\ y'(0) = 0$$

(Consult the attached table of Laplace transforms)

(b) If $g(t) = \cdot 2\cos t$, solve $(*)$ by method of judicious guessing.

5. (15 points)
Let n be a positive integer. Consider the Bessel equation of order n.

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0$$

(a) Show that $x = 0$ is a regular singular point and the root of the indicial equation is n and $-n$.

(b) Corresponding to the larger root n, show that one solution is

$$J_n(x) = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{1}{2}x\right)^{n+2j}}{j!(n+j)!}$$

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TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. t^n, n = positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, n = positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$