國 立 清 華 大 學 命 題 紙

八十六學年度 數學系 系(所) 應數 組碩士班研究生入學考試

科目 高等微積分 科號 0201 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

ADVANCED CALCULUS (97 AP MASTER ENTRANCE EXAM)

1. (20分) Prove or disprove the following statments:
   (a) Let $D = [0,1] \times [0,1]$, $f(x,y) = x^2 + xy + y^2$ and $\Gamma_f = \{(x,y,f(x,y)) : (x,y) \in D\}$. Then Γ_f is connected.
   (b) Suppose that both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on $\mathbf{R}^2$. Then f is continuous on $\mathbf{R}^2$

2. (15分) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be of class C^2 and

$$f(2,1) = 0, \quad \frac{\partial f}{\partial x}(2,1) = 1, \quad \frac{\partial f}{\partial y}(2,1) = -1;$$

$$\frac{\partial^2 f}{\partial^2 x}(2,1) = 2, \quad \frac{\partial^2 f}{\partial x \partial y}(2,1) = -1, \quad \frac{\partial^2 f}{\partial^2 y}(2,1) = -2.$$

   Set $u(x,y) = f(x^2+y^2, xy)$. Find $\frac{\partial^2 u}{\partial y^2}(1,1)$.

3. (20分) Let $f : [0,1] \to \mathbf{R}$ be continuous and

$$a_n = (\log n)^{-2} \int_0^1 t^n f(t)\, dt.$$

   (a) Prove that there exists $M > 0$ such that

$$|a_n| \le \frac{M}{(n+1)(\log n)^2} \qquad \text{for all} \quad n \ge 2.$$

   (b) Does the series $\sum_{n=2}^{\infty} a_n x^n$ converge uniformly on $[-1,1]$? Verify your answer.

4. (15分) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be of class C^1. If $|\nabla f(x,y)| \le 1$ on $\mathbf{R}^2$, prove that

$$\lim_{x \to +\infty} x^{-p} f(x,x) = 0 \quad \text{for all} \quad p > 1.$$

5. (20分)
   (a) Let $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\phi$. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}$.
   (b) Evaluate the triple integral $\iiint_\Omega (x^2+y^2+z^2)^{3/2}\, dxdydz$, where $\Omega = \{(x,y,z) \in \mathbf{R}^3 : 1/2 \le \sqrt{x^2+y^2+z^2} \le 1\}$.

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2 ADVANCED CALCULUS (97 AP MASTER ENTRANCE EXAM)

6. (15分) Evaluate the line integral

$$\oint_C y^3 dx - x^3 dy,$$

where C is the square $|x| + |y| = 1$ in a counterclockwise direction.

7. (15分) Suppose that F is a C^∞ function on $\mathbf{R}^3$. Assume $F(0,0,0) = 0, \frac{\partial F}{\partial x}(0,0,0) = \frac{\partial F}{\partial y}(0,0,0) = 0, \frac{\partial F}{\partial z}(0,0,0) \neq 0$ and $\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 - \frac{\partial^2 F}{\partial x \partial x}\frac{\partial^2 F}{\partial y \partial y} > 0$ at $(0,0,0)$.

(a) Show that there is a C^∞ function ϕ defined in a neighborhood of $(0,0)$ where $F(x,y,\phi(x,y)) = 0$.

(b) Show that $(0,0)$ is a critical point of ϕ and decide what type the critical point is. (local maximum, local minimum, or sad dle point).