國 立 清 華 大 學 命 題 紙

八十六學年度 數學系 系（所） 純數 組碩士班研究生入學考試

科目 複變數函數論 科號 0103 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. Let Γ denote the unit circle $|z| = 1$, which is counterclockwisely oriented. Let f(z) denote the polynomial function $z^6 + 5z^3 + z - 2$.
   (1) (3分) Prove that $f(z) \neq 0$ on Γ.
   (2) (8分) Evaluate the integral $\int_{\Gamma} \frac{f'(z)}{f(z)} dz$.

2. Let C_2 denote the circle $|z| = 2$, which is counterclockwisely oriented. Evaluate the following integrals:
   (1) (8分) $\frac{1}{2\pi i} \int_{C_2} \frac{5z^5}{z^5 - 1} dz$;
   (2) (8分) $\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 6} dx$.

3. (8分) Let D denote the disk $\{z \in C : |z| < 1\}$. Suppose f(z) and g(z) are functions which are holomorphic on D, such that f(z)g(z) = 0 for all z in D. Prove that either f(z) = 0 for all z in D, or g(z) = 0 for all z in D.

4. Let h(z) be a holomorphic function defined on the disk $D_2 = \{z \in C : |z| < 2\}$. Let Γ denote the level set $|h(z)| = 1$ in D_2.
   (1) (7分) Suppose Γ is a simple closed curve in D_2, which encloses a nonempty open subregion Ω_1 in D_2. Prove that the equation
   $$h(z) = 1 + i,$$
   where $i = \sqrt{-1}$, has no solution in Ω_1.
   (2) (10分) Suppose Γ is the unit circle $|z| = 1$, z = 0 is the only zero of h(z) in $|z| < 1$, and z = 0 is a simple zero of h(z). Write $h(z) = zk(z)$. Prove that $k(z)$ is a constant function.

5. (8分) Suppose $f(z)$ is an entire function such that
$$\left|\frac{d^5}{dz^5} f(z)\right| \leq 1$$
for all z in the complex plane. Prove that
$$\int_0^{2\pi} f(a - e^{i\theta}) e^{-6i\theta} d\theta = 0$$
for any complex number a.