國 立 清 華 大 學 命 題 紙

八十六學年度 數學系 系（所） 純數 組碩士班研究生入學考試

科目 代數及線性代數 科號 0102 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

Abstract and Linear Algebra(0102)

1.(16 points) Denote a point in R^4 by (x_1, x_2, x_3, x_4), let P be the plane in R^4 determined by $x_1 + x_2 + x_3 = 0$, $x_1 - x_2 + x_4 = 0$.

(a) Find the orthogonal projection A from R^4 onto P.

(b) Find $trace(A^{10})$.

2.(14 points) Let A denote the reflection on R^3 with respect to the line $x = y = z$, B denote the rotation of 180° on R^3 with z-axis as its rotation axis. Find AB and give the geometric meaning.

3.(14 points) Given real matrices $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Are A, B similar? Prove your answer.

4.(16 points) Let V be a finite dimensional inner product space; $A, B : V \to V$ be linear transformations. Prove that

(a) $Ker A \subset Ker B$ implies that $B = PA$ for some linear transformation P on V.

(b) $\|Ax\| = \|Bx\|$ for all $x \in V$ implies that $B = PA$ for some isometry P on V.

5.(15 points) Consider the polynomial $f(t) = t^4 + 15t^3 + 22 \in Z[t]$. For a prime number p, denote by $\pi : Z \to Z_p$ the canonical projection onto the field of p elements, and $\bar{f}(t) = t^4 + \pi(15)t^3 + \pi(22) \in Z_p[t]$.

(a) Show that $\bar{f}$ is reducible in $Z_3[t]$.

(b) Show that $\bar{f}$ is irreducible in $Z_5[t]$.

(c) Show that f is irreducible in $Q[t]$, Q is the field of rational numbers.

6.(10 points) Let G be a group of order 15, show that G is a cyclic group.

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7.(20 points) Let $k = Z_3$ be the field consisting of 3 elements,

$G = Gl(2,k) = \{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} ; a,b,c,d \in k, ad - bc \neq 0\}$

(a) Show that there are exactly 4 distinct lines in k^2 passing through $(0,0)$, denote them by $\ell_1, \ell_2, \ell_3, \ell_4$.

(b) Each $g \in G$ induces a bijective map on $\{\ell_1, \ell_2, \ell_3, \ell_4\}$. Show that this gives a homomorphism $f : G \to S_4$.

(c) Find the kernel and the image of f respectively.

(d) Find the order of G.

8.(15 points) Regard $R = Z[\sqrt{-1}] = \{m + n\sqrt{-1}; m, n \text{ integers}\}$ as a subring of the complex number field, $I = (1 + 3\sqrt{-1})$ the principal ideal of R generated by $1 + 3\sqrt{-1}$.

(a) Show that we can find a positive integer n so that R/I is (ring) isomorphic to Z/nZ.

(b) Is I a prime ideal in R? Why?