國 立 清 華 大 學 命 題 紙

八十五學年度 數學 系(所) 純數 組碩士班研究生入學考試

科目 複變數函數論 科號 0103 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (21%)

Evaluate the following integrals:

(a) $\int_0^{2\pi} \frac{dx}{(2+\cos x)^2}$ (b) $\int_0^{\infty} \frac{\sin^2 2x}{x^2} dx$ (c) $\int_0^{\infty} \frac{dx}{1+x^4}$

2. (8%)

Let $f(z)$ be a continuous function on $U = \{z : |z| < 1\}$. If $f(z)$ is holomorphic at every point $z = x + iy$ when $xy \neq 0$. Does it imply $f(z)$ is holomorphic on U. Prove it or give a counter example.

3. (9%)

Find a harmonic function on $U = \{z : |z| < 1\}$ such that

$$\lim_{z \to \zeta} f(z) = \begin{cases} 1 & \text{if } |\zeta| = 1 \text{ and } \operatorname{Im} \zeta > 0 \\ 0 & \text{if } |\zeta| = 1 \text{ and } \operatorname{Im} \zeta < 0. \end{cases}$$

(Hint: It is easier to do it on the upper half plane)

4. (8%)

Let $f(z)$ be a holomorphic function on $U = \{z : |z| < 1\}$ with real part $u(x, y)$ and imaginary part $v(x, y)$. Let $h(\xi, \eta)$ be a continuous differentiable function on $\mathbb{R}^2$ such that $\nabla h(\xi, \eta) \neq 0$ whenever $h(\xi, \eta) = 0$. If $h(u(x, y), v(x, y)) \equiv 0$, show that $f(z)$ is a constant function. (Hint: Cauchy-Riemann equation)

5. (14%)

Let $f_n(z), n = 1, 2, \cdots$ be a sequence of holomorphic function on $U : \{z : |z| < 1\}$. Assume $\{f_n(z)\}$ converges uniformly to $f(z)$ on every $U_r = \{z : |z| < r\}, 0 < r < 1$.

(a) Show that $f(z)$ is holomorphic in U.

(b) If $f_n(z) \neq 0$ on U for all $n = 1, 2, \cdots$,

Show that either $f(z) \equiv 0$ or $f(z) \neq 0$ on U.

(Hint: Rouchè's theorem is helpful)