國立清華大學命題紙

八十四學年度 應用數學 所 ______ 組碩士班研究生入學考試

科目 數值分析 科號 0204 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (a) Suppose that we want to use computer to compute the value of $\sqrt{1+2x}-\sqrt{1+x}$ for x near 0. Get one computing method to avoid loss of significance. (8 points)

   (b) Let $p(x)=x^3+ax^2+bx+c$ for some constants a, b, c. Suppose we know that there exists a x_0 with large magnitude such that $p(x_0)$ has very small magnitude. Explain why we can not compute $p(x_0)$ with high accuracy by computer in general. (8 points)

2. Consider fixed-point iteration $x_{n+1}=g(x_n)$ for some given point x_0. Here g is a smooth function on the whole real line. Suppose that there exists a ξ such that $\xi=g(\xi)$ and $g'(\xi)=g''(\xi)=\cdots=g^{(n)}(\xi)=0$ for some positive integer $n\geq 1$.

   (a) Prove that there is a number $\varepsilon>0$ such that if $|x_0-\xi|\leq\epsilon$, then $x_n\to\xi$ as $n\to\infty$. (8 points)

   (b) Let x_0 be a point which satisfies the condition of part (a). We assume that $x_n\neq\xi$ for all n. Find a constant C and a positive integer p such that

$$\lim_{n\to\infty}\frac{|x_{n+1}-\xi|}{|x_n-\xi|^p}=C. \qquad \text{(8 points)}$$

3. (a) Suppose we want to use the integration quadrature rule $Q(f)=A_0f\left(\frac{1}{4}\right)+A_1f\left(\frac{1}{2}\right)+A_2f\left(\frac{3}{4}\right)$ to approximate the integral $I(f)=\int_0^1 f(x)dx$. Here A_0, A_1, A_2 are constants independent of f. Find the constants A_0, A_1, A_2 such that $Q(f)=I(f)$ for all polynomials f of degree less than or equal to 2. (8 points)

   (b) Given two real numbers a and b with $a<b$ and $x_i=a+i\cdot\frac{b-a}{N}$ for $i=0,1,\cdots,N$. Here N is a positive integer. Set $h=(b-a)/N$. Let f be a smooth function on $[a,b]$. Denote the composite midpoint integration rule by $M_N=h\sum_{i=1}^{N}f\left(\frac{x_{i-1}+x_i}{2}\right)$. Then there exist constants $C_1, C_2, \cdots$, independent of h such that $I(f)=\int_a^b f(x)dx=M_N(f)+C_1h^2+C_2h^4+O(h^6)$. Find the constant C_1. (10 points)

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4. (a) Let A be an $n \times n$ symmetric and positive deifnite matrix. Use mathematical induction to prove that there exists an $n \times n$ lower triangular matrix L such that $A = LL^T$. (10 points)

   (b) Let A be a general $n \times n$ matrix. Suppose we can find an $n \times n$ lower triangular matrix L with $L_{ii} = 1$, $\forall\, i$, and $|L_{ij}| \leq 1$, $\forall\, i, j$ and an $n \times n$ upper triangular matrix U such that $A = LU$. Use mathematical induction to prove that $\max_{i,j} |U_{ij}| \leq 2^{n-1} \cdot \max_{i,j} |A_{ij}|$. (10 points)