注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 109 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 作答中如有發現試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 109 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

　　*請在【答案卷、卡】作答

1. (10 pts) Let $f:(0,\infty)\to\mathbb{R}$ be differentiable. If $f(x)\to 5$ and $f'(x)\to\lambda$ as $x\to\infty$, prove that $\lambda=0$.

2. (10 pts) Find the limit
$$\lim_{n\to\infty}\frac{n}{\log n}\left(n^{1/n}-1\right).$$

3. (12 pts) Prove that the function
$$f(x)=\sum_{n=1}^{\infty}\frac{x^2}{x^2+n^2}$$
is continuous on $\mathbb{R}$.

4. (12 pts) Suppose that $\{p_n\}$ is a sequence of polynomials, and that $p_n\to f$ uniformly on the interval $[0,1]$. Must f be differentiable?

5. (12 pts) Is there a simple closed curve C in the xy-plane which maximizes the value of
$$\oint_C y^3\,dx+\left(3x-x^3\right)dy\ ?$$
If so, find the maximum value.

6. (12 pts) Let $B=\{x\in\mathbb{R}^n/\,\|x\|\le r\}$, and suppose $f:B\to\mathbb{R}^n$ satisfying $\|f(0)\|\le\frac{2}{3}r$ and
$$\|f(x)-f(y)\|\le\frac{1}{3}\|x-y\|\quad\text{for all } x,y\in B.$$
Prove that there exists a unique $x\in B$ such that $f(x)=x$.

7. (16 pts) Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be defined by $F(x)=L(x)+G(x)$, where L is a linear isomorphism and G is a C^1-function. Suppose that there are positive constants M and ε such that $\|G(x)\|\le M\|x\|^{1+\varepsilon}$ for all x in a neighborhood of the origin. Prove that F is locally invertible near the origin.

8. (16 pts) Let $f:(a,b)\to\mathbb{R}$ be a C^n-function, and suppose for some $c\in(a,b)$,
$$f'(c)=f''(c)=\cdots=f^{(n-1)}(c)=0,\qquad\text{but}\quad f^{(n)}(c)\ne 0.$$
Prove that

   (a) For n even, f has a local minimum at c if $f^{(n)}(c)>0$, and a local maximum at c if $f^{(n)}(c)<0$.

   (b) For n odd, there is neither a local maximum nor a local minimum at c.