注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 108 學年度碩士班考試入學試題

系所班組別：數學系

考試科目(代碼)：線性代數(0102)

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 作答中如有發現試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 108 學年度碩士班考試入學試題

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 *請在【答案卷、卡】作答

1 (12%) Let $f : V \to W$ and $g : W \to Z$ be linear transformations on finite-dimensional real vector spaces V, W and Z. Show that:

(a) $\operatorname{rank}(g \circ f) \leq \operatorname{rank}(g)$.

(b) $\operatorname{rank}(g \circ f) \leq \operatorname{rank}(f)$.

2 (12%) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

3 (12%) Suppose that $1, 2, \ldots, n$ are eigenvalues of a nonzero linear transformation T on a finite-dimensional vector space V over $\mathbb{C}$. Let $v_k \neq 0$ be an eigenvector of T with eigenvalue k for $k = 1, \ldots, n$. Show that $v_1, \ldots, v_n$ are linearly independent over $\mathbb{C}$.

4 (12%) Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be an invertible matrix for which $A^* = A^{-1}$, where A^* is the conjugate transpose of A. Let $f(X) \in \mathbb{C}[X]$ be the characteristic polynomial of A. Show that

$$f(X) = \prod_{k=1}^{n} (e^{i\theta_k} - X)$$

for some $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

5 (12%) Let $A, B \in \mathrm{Mat}_n(\mathbb{C})$ be two invertible matrices. Show that the matrix $A + \lambda B$ is invertible for all but finitely many $\lambda \in \mathbb{C}$.

6 (12%) Let $A, B \in \mathrm{Mat}_n(\mathbb{C})$. Show that every eigenvalue of AB is also an eigenvalue of BA.

7 (14%) Let V be a complex vector space of finite dimension with a fixed positive definite hermitian inner product $<, >$. Let $f : V \to V$ be a linear transformation over $\mathbb{C}$ such that $< f(v), v >= 0$. Show that f is a zero map.

8 (14%) Let A be an $n \times n$ real matrix and A^{tr} be its transpose. Show that A^{tr} and $A^{tr}A$ have the same rank.