國立清華大學 107 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

　　*請在【答案卷、卡】作答

1. [8%] Let W be (real) space of all 2×2 complex Hermitian matrices, i.e., the space of all 2×2 complex matrices A such that $A^t = \overline{A}$ where A^t denotes the transpose of A, and $\overline{[a_{i,j}]} = [\overline{a_{i,j}}]$. Show that the mapping

$$(a, b, c, d) \mapsto \begin{bmatrix} a+d & b+ci \\ b-ci & a-d \end{bmatrix}$$

is a vector space isomorphism from $\mathbb{R}^4$ onto W where $i = \sqrt{-1}$.

2. [8%] Find the value of k that satisfies the equation:

$$\det \begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} = k \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

3. [8%] Find the rank of the matrix

$$A = \begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}.$$

4. [8%] Find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 0 & 4 & 2 \\ -1 & -4 & -1 \\ 0 & 0 & -2 \end{bmatrix}.$$

5. [10%] Consider the vectors

$$\mathbf{v}_1 = (3, 0, 4)$$
$$\mathbf{v}_2 = (-1, 0, 7)$$
$$\mathbf{v}_3 = (2, 9, 11)$$

in $\mathbb{R}^3$ equipped with the standard inner product. Find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbb{R}^3$ such that

$$\text{Span}\{\mathbf{w}_1\} = \text{Span}\{\mathbf{v}_1\};$$
$$\text{Span}\{\mathbf{w}_1, \mathbf{w}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\};$$
$$\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

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6. [10%] A complex $n \times n$ matrix X is called *nilpotent* if $X^m = 0$ for some positive integer m. Let A be a nilpotent $n \times n$ complex matrix. Show that $\mathrm{Tr}(A^r) = 0$ for all positive integer r.

7. [16%] Let $(V, \langle , \rangle)$ be a 4-dimensional real inner product space with an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Let W be a 3-dimensional subspace spanned by an orthonormal set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ (i.e., $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_{i,j}$ for any i, j). Let $\pi \colon V \to W$ be the orthogonal projection. It is known that

$$\begin{aligned}
\pi(\mathbf{v}_1) &= c(\ \ \mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3);\\
\pi(\mathbf{v}_2) &= c(\ \ \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3);\\
\pi(\mathbf{v}_3) &= c(-\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3);\\
\pi(\mathbf{v}_4) &= c(-\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3)
\end{aligned}$$

for some positive constant c.

(1) Find the kernel of π.

(2) Find c.

8. [16%] Let A be a symmetric $n \times n$ real matrix. Show that the following two conditions are equivalent:

(1) All eigenvalues of A are positive.

(2) $\mathbf{x}^t A \mathbf{x} > 0$ for all $n \times 1$ real column vectors $\mathbf{x}$.

9. [16%] Let V be a finite-dimensional vector space over a field F, and let V^* denote the dual space of V. For $v \in V$, define $\xi_v \in (V^*)^*$ by $\xi_v(f) = f(v)$ where $f \in V^*$.

(1) Show that the map $\Xi \colon V \to (V^*)^*$ given by $v \mapsto \xi_v$ is a vector space isomorphism.

(2) For a subspace W of V, define $W^\perp = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}$. Show that $(W^\perp)^\perp = \Xi(W)$ where Ξ is given in (1).