國立清華大學 107 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：高等微積分（0101）

　　*請在【答案卷、卡】作答

1. Let

$$F(x) = \int_1^{x^2} \frac{\sin t}{2 + e^t} dt$$

   (a) (10%) Determine all local maximum and local minimum of F.

   (b) (10%) Show that $|F(x)| \le |x - 1|$ for all $x \in \mathbb{R}$.

2. Let $[x]$ denote the greatest integer less than or equal to x.

   (a) (5%) Is the function $[\ \ ]$ Riemann integrable on the interval $[1, 10]$? Why?

   (b) (10%) Evaluate the double integral

$$\iint_R [x + y] dA$$

   where

$$R = \{(x, y) | 0 \le x \le 2, 4 \le y \le 6\}$$

3. Let $A, B \subset \mathbb{R}^3$ be two nonempty subsets.

   (a) (10%) Show that if A and B are compact and for any $n \in \mathbb{N}$, there exist $a_n \in A, b_n \in B$ such that $|a_n - b_n| < \frac{1}{n}$ where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^3$. Show that A and B are not disjoint.

   (b) (5%) If A and B are not compact, is the above result true? Prove or give a counterexample.

4. (10%) Let $E : \mathbb{R}^3 - \{(0, 0, 0)\} \to \mathbb{R}^3$ be defined by

$$E(\mathbb{X}) = \frac{1}{|\mathbb{X}|^3} \mathbb{X}$$

   where $\mathbb{X} = (x, y, z), |\mathbb{X}| = \sqrt{x^2 + y^2 + z^2}$. Suppose that S is a closed smooth surface that encloses the origin, show that the surface integral of E over S is

$$\iint_S E \cdot dS = 4\pi$$

5. (20%)Let $\mathscr{C}(\mathbb{R}^2) := \{f : \mathbb{R}^2 \to \mathbb{R} | f \text{ is continuous }\}$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function.

   (a) Let
   $$C_r := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = r^2\}$$
   Suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on C_r, for each $r \geq 0$. Does it imply that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $\mathbb{R}^2$?

   (b) If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on each compact subset of $\mathbb{R}^2$, does it imply that $f \in \mathscr{C}(\mathbb{R}^2)$? Prove or give a counterexample.

6. (10%) Let $f : [2, 7] \to \mathbb{R}$ be a continuous function. Given $\varepsilon > 0$. Show that there is a polynomial p such that
   $$p(2) = f(2), \quad p'(2) = 0, \quad \text{and } \sup\{|p(x) - f(x)| | x \in [2, 7]\} < \epsilon$$

7. (10%) Let (X, d) be a ***compact*** metric space and $f : X \to X$ be a function which satisfies the following condition
   $$d(f(x), f(y)) < d(x, y)$$
   for all $x, y \in X$. Show that f has a unique fixed point.