1. **(10%)** Let $W$ be a subspace of a vector space $V$. Show that if $\beta$ is a basis for $W$, and $v \in V \setminus \beta$, then $\beta \cup \{v\}$ is linearly independent if and only if $v \notin W$.

2. **(12%)** Let $V$ and $W$ be finite-dimensional vector spaces, and let $T : V \to W$ be a surjective linear transformation. Show that if $U$ is a subspace of $V$, then

$$\dim T(U) \geq \dim U - \dim V + \dim W.$$

3. **(12%)** Let $A$ be the coefficient matrix of a homogeneous system of $m$ linear equations in $n$ unknowns $x_1, \ldots, x_n$. Show that if we include one more equation

$$c_1 x_1 + \cdots + c_n x_n = 0$$

in the system, the dimension of the solution space will decrease by one if the vector $(c_1, \ldots, c_n)$ cannot be spanned by the row vectors of $A$.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation by $30°$ about the axis spanned by the vector $(1, 2, 3)$. The rotation is counter-clockwise when the vector $(1, 2, 3)$ points toward the observer. Let $A$ be the matrix representation of $T$ relative to the standard basis for $\mathbb{R}^3$, and let $A^t$ denote the transpose of $A$.

   (a) **(10%)** Find $A^{12}$.

   (b) **(10%)** Find all integers $n$ such that $A^n = (A^t)^n$.

   (c) **(12%)** Find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1} A^6 Q = D$.

5. **(12%)** Show that for any invertible linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$, there exists a line $L$ in $\mathbb{R}^3$ passing through the origin such that $T(L) = L$.

6. **(12%)** Let $A$, $B$ and $Q$ be square matrices with real entries. Show that if $A$ is symmetric, $B$ is skew-symmetric, and $Q$ is invertible such that $Q^{-1} A Q = B$, then $A = B = 0$.

7. **(10%)** Let $f$ be the piecewise continuous function on the interval $[-1, 1]$ such that $f(x) = 0$ for $x \in [-1, 0)$ and $f(x) = 1$ for $x \in [0, 1]$. Find the polynomial function $g$ of degree $\leq 2$ which best approximates $f$, in the sense that the value of the integral $\int_{-1}^{1} |f(x) - g(x)|^2 \, dx$ is minimized.