1. (24 pts) Determine which of the following statements are true. Give reasons for your answers.

   (a) The set $A = \left\{ x \in [-2, 2] : x \cos x \geq \frac{1}{2} \sin x \right\}$ is compact.

   (b) The sequence $\{\sin n\}_{n=1}^{\infty}$ contains a convergent subsequence.

   (c) There is a continuous one-to-one map taking the unit circle onto the interval $[0, 1]$.

2. (8 pts) If $\{a_n\}$ is a sequence such that $|a_{n+1} - a_n| \leq \frac{1}{2^n}$, prove that $\{a_n\}$ is convergent.

3. (8 pts) Prove that the function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

4. (8 pts) Find $\limsup_{n \to \infty} a_n$ and $\liminf_{n \to \infty} a_n$ if $a_n = \left(1 + \frac{1}{n}\right) \cos \frac{n\pi}{3}$.

5. (8 pts) Let

$$\alpha(x) = \begin{cases} 0 & \text{if} \quad x \leq 1 \\ \frac{1}{2} & \text{if} \quad 1 < x \leq 2 \\ 2 & \text{if} \quad x > 2 \end{cases}.$$

Evaluate the Riemann-Stieltjes integral $\int_0^3 \log(x+1) \, d\alpha(x)$.

6. (10 pts) Let $f, g : [a, b] \to \mathbb{R}$. If g is continuous, f is Riemann-integrable and $f \geq 0$, prove that there is some $c \in [a, b]$ such that

$$\int_a^b f(x) g(x) \, dx = g(c) \int_a^b f(x) \, dx.$$

7. (12 pts) Let $f_n(x) = nx \left(1 - x^2\right)^n$ for $0 \leq x \leq 1$ and $n = 1, 2, \cdots$.

   (a) Find $\lim_{n \to \infty} f_n(x)$ for $0 \leq x \leq 1$.

   (b) Does $\{f_n\}$ converge uniformly on $[0, 1]$?

8. (12 pts)

   (a) Expand $f(x) = \frac{x}{(1-x)^2}$ as a power series.

   (b) Use (a) to find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$.

9. (10 pts) Let $E \subset \mathbb{R}^n$ be open. If $f : E \to \mathbb{R}^n$ is a C^1 mapping and if $Df(x)$ is invertible for every $x \in E$, prove that f is an open mapping; that is, $f(W)$ is an open subset of $\mathbb{R}^n$ for every open set $W \subset E$.