國立清華大學 106 學年度碩士班考試入學試題

系所班組別：數學系碩士班 數學組

考試科目（代碼）：代數與線性代數（0102）

 *請在【答案卷、卡】作答

1. **(10%)** Let W be a subspace of a vector space V. Show that if β is a basis for W, and $v \in V \setminus \beta$, then $\beta \cup \{v\}$ is linearly independent if and only if $v \notin W$.

2. **(12%)** Determine if the symmetric group S_4 has an abelian subgroup of order 6, and prove your answer.

3. **(10%)** Show that the factor ring $\mathbb{Z}[x]/\langle x^2+1\rangle$ is isomorphic to the ring of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$.

4. **(12%)** In the ring of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$, determine if the ideal
$$I = \langle 7+5\sqrt{-1}, 8-2\sqrt{-1}\rangle$$
is principal, and if so, find a single element that generates I.

5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation by 30° about the axis spanned by the vector $(1,2,3)$. The rotation is counter-clockwise when the vector $(1,2,3)$ points toward the observer. Let A be the matrix representation of T relative to the standard basis for $\mathbb{R}^3$, and let A^t denote the transpose of A.
   (a) **(10%)** Find all integers n such that $A^n = (A^t)^n$.
   (b) **(12%)** Find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}A^6Q = D$.

6. **(12%)** Show that for any invertible linear operator $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, there exists a line L in $\mathbb{R}^3$ passing through the origin such that $T(L) = L$.

7. **(12%)** Let A, B and Q be square matrices with real entries. Show that if A is symmetric, B is skew-symmetric, and Q is invertible such that $Q^{-1}AQ = B$, then $A = B = 0$.

8. **(10%)** Let $a = \dfrac{1+\sqrt[3]{2+\sqrt{3}}}{4+\sqrt[5]{5}}$. Show that there exists a nonzero polynomial f of degree ≤ 30 with rational coefficients such that $f(a) = 0$.