國立清華大學 106 學年度碩士班考試入學試題

系所班組別：數學系碩士班 數學組

考試科目（代碼）：高等微積分（0101）

共__1__頁，第__1__頁　　*請在【答案卷、卡】作答

1. (24 pts) Determine which of the following statements are true. Give reasons for your answers.
   - (a) The set $A = \left\{x \in [-2, 2] : x \cos x \geq \frac{1}{2} \sin x\right\}$ is compact.
   - (b) The sequence $\{\sin n\}_{n=1}^{\infty}$ contains a convergent subsequence.
   - (c) There is a continuous one-to-one map taking the unit circle onto the interval $[0, 1]$.

2. (8 pts) If $\{a_n\}$ is a sequence such that $|a_{n+1} - a_n| \leq \frac{1}{2^n}$, prove that $\{a_n\}$ is convergent.

3. (8 pts) Prove that the function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

4. (8 pts) Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ if $a_n = \left(1 + \frac{1}{n}\right) \cos \frac{n\pi}{3}$.

5. (8 pts) Let

$$\alpha(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \frac{1}{2} & \text{if } 1 < x \leq 2 \\ 2 & \text{if } x > 2 \end{cases}.$$

   Evaluate the Riemann-Stieltjes integral $\int_0^3 \log(x+1)\, d\alpha(x)$.

6. (10 pts) Let $f, g : [a, b] \to \mathbb{R}$. If g is continuous, f is Riemann-integrable and $f \geq 0$, prove that there is some $c \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = g(c) \int_a^b f(x)\, dx.$$

7. (12 pts) Let $f_n(x) = nx\left(1 - x^2\right)^n$ for $0 \leq x \leq 1$ and $n = 1, 2, \cdots$.
   - (a) Find $\lim_{n\to\infty} f_n(x)$ for $0 \leq x \leq 1$.
   - (b) Does $\{f_n\}$ converge uniformly on $[0, 1]$?

8. (12 pts)
   - (a) Expand $f(x) = \frac{x}{(1-x)^2}$ as a power series.
   - (b) Use (a) to find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$.

9. (10 pts) Let $E \subset \mathbb{R}^n$ be open. If $f : E \to \mathbb{R}^n$ is a C^1 mapping and if $Df(x)$ is invertible for every $x \in E$, prove that f is an open mapping; that is, $f(W)$ is an open subset of $\mathbb{R}^n$ for every open set $W \subset E$.