國立清華大學 105 學年度碩士班考試入學試題

系所班組別：數學系　應用數學組

考試科目（代碼）：線性代數 (0202)

　　*請在【答案卷】作答

1. (10%) Let $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \le z^2\}$. Is S a vector subspace of $\mathbb{R}^3$? Prove your claim.

2. (10%) Let P be the collection of all real polynomials. Is P a ***finite*** dimensional real vector space under standard polynomial addition and scalar multiplication? Prove your claim.

3. (10%) Suppose that A is an $n \times n$ real matrix satisfying the equation

$$A^3 - 10A + I = 0$$

where I is the $n \times n$ identify matrix. Show that A is invertible.

4. (10%) Show that if λ is the only eigenvalue of a symmetric matrix A, then $A = \lambda I$.

5. (15%) Given an $n \times n$ real matrix $A = [a_{ij}]$ where each a_{ij} is positive. If

$$\sum_{j=1}^{n} a_{ij} = 1$$

for all $i = 1, ..., n$. Show that 1 is an eigenvalue of A and the dimension of the real eigenspace of 1 is 1.

6. (15%) Suppose that $\mathbb{X}(t) = (x_1(t), x_2(t))$ is a vector-valued function of t. Write $\frac{d\mathbb{X}}{dt} = (\frac{dx_1}{dt}, \frac{dx_2}{dt})$. Let

$$A = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$$

Solve the second-order differential equation

$$\frac{d^2\mathbb{X}}{dt^2} = A\mathbb{X}$$

7. (15%) Let

$$\mathscr{A} = \{ae^x \sin x + be^x \cos x + ce^x | a, b, c \in \mathbb{C}\}$$

be the complex vector space generated by the 3 functions $e^x \sin x, e^x \cos x, e^x$. Define $T : \mathscr{A} \to \mathscr{A}$ by

$$T(f) = f'$$

the derivative of f. Find all complex eigenvalues of T and corresponding eigenspaces.

8. (15%) A vector $\mathbb{X} \in \mathbb{R}^n$ is called integral if every component of x_i is an integer. Let A be a nonsingular $n \times n$ matrix with integer entries. Prove that the system of equations $A\mathbb{X} = \mathbb{B}$ has an integral solution for every integral vector $\mathbb{B} \in \mathbb{R}^n$ if and only if $\det A = \pm 1$.