國立清華大學 105 學年度碩士班考試入學試題

系所班組別：數學系　數學組

考試科目（代碼）：高等微積分 (0101)

1. Suppose S is a connected set in $\mathbb{R}^2$ that contains $(1, 3)$ and $(4, -1)$. Show that S contains at least one point on the line $x = y$.

2. Let $f : [a, b] \to \mathbb{R}$ is an integrable function. If for every subinterval $I \subset [a, b]$ there is a point $c \in I$ such that $f(c) = -1$. Compute $\int_a^b f(x)\, dx$.

3. Find the absolute maximun and minimum of $f(x, y) = x^2 + y^2 + y$ on the disc $x^2 + y^2 \leq 1$.

4. Let $f_n(x) = g(x)\, x^n$, where g is continous on $[0, 1]$ and $g(1) = 0$. Show that $f_n \to 0$ uniformly on $[0, 1]$.

5. Suppose f is a function defined on an open set $S \subset \mathbb{R}^3$. Show that if the partial derivatives $\frac{\partial f}{\partial x_j}$ exist and are bounded on S, then f is continous on S.

6. Let $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ if $f(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove or disprove f is differentiable at $(0, 0)$.

7. Let f_n be a sequence of functions that converges uniformly to f on $I \subset \mathbb{R}$ and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in I$. If g is continuous on $\mathbb{R}$, show that the sequence $g \circ f_n$ converges uniformly to $g \circ f$ on I.

8. Let f_n, g_n, h_n be sequences of functions on $\mathbb{R}$ that satisfy $f_n \leq g_n \leq h_n$ on $\mathbb{R}$ for all $n \in \mathbb{N}$. If $\sum f_n$ and $\sum h_n$ converge, show that $\sum g_n$ converges.

9. Let $f : [a, b] \to \mathbb{R}$ is a continuous function. If for each $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$, show that there exists a point $c \in [a, b]$ such that $f(c) = 0$.

（第9題12分，其餘每題11分）